Windings of some planar Stochastic Processes and Applications to the rotation of a polymer

Stavros Vakeroudis

Ecole Doctorale de Paris Centre

Thèse de doctorat

pour l'obtention du titre de

Docteur de l'Université Pierre et Marie Curie (PARIS VI)

Spécialité : Mathématiques

présentée par

Stavros VAKEROUDIS

Titre

Nombres de tours de certains processus stochastiques plans et applications à la rotation d'un polymère

Sous la direction de David HOLCMAN et Marc YOR

Soutenue le 6 avril 2011, devant le jury composé de:

M. Philippe BOUGEROL	Université PARIS VI	Président
M. Alain COMTET	Université PARIS VI	Rapporteur
M. Jacques FRANCHI	Université de STRASBOURG	Rapporteur
M. David HOLCMAN	Ecole Normale Supérieure	Directeur de thèse
M. Yves LE JAN	Université PARIS XI	Examinateur
M. Khashayar PAKDAMAN	Université PARIS VII	Examinateur
M. Zhan SHI	Université PARIS VI	Examinateur
M. Marc YOR	Université PARIS VI	Directeur de thèse

To my family . . .

Στην οικογένειά μου . . .

Résumé

Dans cette thèse de Doctorat, on étudie dans un premier temps les processus d'Ornstein-Uhlenbeck à valeurs complexes $(Z_t = X_t + iY_t, t \geq 0)$, où $(X_t, t \geq 0)$ et $(Y_t, t \geq 0)$ sont ses coordonnées cartésiennes. En prenant le paramètre du processus d'Ornstein-Uhlenbeck égal à 0, on discute, en particulier, le cas du mouvement brownien plan. Plus précisément, on étudie la distribution de certains temps d'atteinte associés aux nombres de tours autour d'un point fixé. Pour obtenir des résultats analytiques, on utilise et on étend l'identité de Bougerol. Cette identité dit que, pour un mouvement brownien réel $(\beta_t, t \geq 0)$, pour tout $u > 0$ fixé,

$$\sinh(\beta_u) \overset{(loi)}{=} \hat{\beta}_{(\int_0^u ds \exp(2\beta_s))},$$

où $(\hat{\beta}_t, t \geq 0)$ est un mouvement brownien, indépendant de β.

Nous développons quelques identités en loi concernant les processus d'Ornstein-Uhlenbeck à valeurs complexes, qui sont équivalentes à l'identité de Bougerol. Ces identités nous permettent de caractériser les lois de temps d'atteinte $T_c^\theta \equiv \inf\{t : \theta_t = c\}$, $(c > 0)$ du processus continu des nombres de tours associé au processus d'Ornstein-Uhlenbeck à valeurs complexes Z, défini par $\theta_t \equiv \mathrm{Im}(\int_0^t \frac{dZ_s}{Z_s}), t \geq 0$. De plus, on étudie la distribution du temps aléatoire $T_{-d,c}^\theta \equiv \inf\{t : \theta_t \notin (-d, c)\}$, $(c, d > 0)$ et particulièrement de $T_{-c,c}^\theta \equiv \inf\{t : \theta_t \notin (-c, c)\}$, $(c > 0)$.

Une étude approfondie de l'identité de Bougerol montre que $1/A_u(\beta)$, où $A_u(\beta) = \int_0^u ds \exp(2\beta_s)$, considéré sous une mesure appropriée, changée à partir de la mesure de Wiener, est infiniment divisible.

En utilisant les résultats précédents, on estime le temps de rotation moyen, noté TRM. Ce dernier est la moyenne du premier temps pour qu'un polymère plan modélisé comme une collection de n cordes paramétrées par un angle brownien fasse un tour autour d'un autre point (winding). On est ainsi conduit à étudier une somme d'exponentielles i.i.d. avec mouvements browniens réels en arguments. On montre que la position finale satisfait à une nouvelle équation stochastique, avec un drift non-linéaire. Finalement, on obtient une formule asymptotique pour le TRM. Le terme dominant dépend de $\sqrt{n}$ et, notablement, il dépend aussi faiblement de la configuration initiale moyenne. Nos résultats analytiques sont confirmés par des simulations browniennes.

Abstract

In this PhD thesis, we first study the (planar) complex valued Ornstein-Uhlenbeck processes $(Z_t = X_t + iY_t, t \geq 0)$, where $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ denote its cartesian coordinates. Taking the Ornstein-Uhlenbeck parameter equal to 0 allows to discuss in particular the planar Brownian motion case. More precisely, we study the distribution of several first hitting times related to the winding process around a fixed point. To obtain analytical results, we use and extend Bougerol's identity. This identity says that for a linear Brownian motion $(\beta_t, t \geq 0)$, for any fixed $u > 0$,

$$\sinh(\beta_u) \stackrel{(law)}{=} \hat{\beta}_{(\int_0^u ds \exp(2\beta_s))},$$

with $(\hat{\beta}_t, t \geq 0)$ a Brownian motion, independent of β.

We develop some identities in law in terms of (planar) complex valued Ornstein-Uhlenbeck processes, which are equivalent to Bougerol's identity. This allows us to characterize the laws of the random hitting times $T_c^\theta \equiv \inf\{\ t : \theta_t = c\ \}, \quad (c > 0)$ of the continuous winding processes $\theta_t \equiv \mathrm{Im}(\int_0^t \frac{dZ_s}{Z_s}), t \geq 0$ associated with our complex Ornstein-Uhlenbeck process. Moreover, we investigate the distribution of the random time $T_{-d,c}^\theta \equiv \inf\{\ t : \theta_t \notin (-d, c)\ \}, \quad (c\ ,\ d > 0)$ and, more specifically of $T_{-c,c}^\theta \equiv \inf\{\ t : \theta_t \notin (-c, c)\ \}, \quad (c > 0)$.

We investigate further Bougerol's identity, and we show that $1/A_u(\beta)$, where $A_u(\beta) = \int_0^u ds \exp(2\beta_s)$, considered after a suitable measure change from Wiener measure, is infinitely divisible.

Using the previous results, we estimate the mean rotation time (MRT) which is the mean of the first time for a planar polymer, modeled as a collection of n rods parameterized by a Brownian angle, to wind around a point. We are led to study the sum of i.i.d. exponentials with one dimensional Brownian motions as arguments. We find that the free end of the polymer satisfies a novel stochastic equation with a nonlinear time function. Finally, we obtain an asymptotic formula for the MRT, and the leading order term depends on $\sqrt{n}$ and, interestingly, it also depends weakly upon the mean initial configuration. Our analytical results are confirmed by Brownian simulations.

Acknowledgements

During the last few years, I have realized that conducting PhD research is not an easy task and it would have never been possible if it was not for my advisors Professor David Holcman and Professor Marc Yor. I feel very grateful to both of them because they taught me how research progresses. Moreover, they offered me the opportunity to profit from their knowledge. David introduced me to the fascinating domain of modern stochastic modeling in Biology, by using a probabilistic approach. His interest and his availability to answer every -at times naive- question that I posed, were constant. I also feel very grateful that i worked with Marc. He was a source of knowledge and inspiration and his way of thinking and researching will always be deep in my mind.

I would also like to express my gratitude to the reviewers of my PhD thesis, Professor Alain Comtet and Professor Jacques Franchi. Their very useful remarks were crucial for the improvement of this manuscript.

Furthermore, I wish to extend my appreciations to the other members of my PhD committee, Professors Philippe Bougerol, Yves Le Jan, Khashayar Pakdaman and Zhan Shi, for the time that they have devoted to my PhD thesis.

I should also thank Professor Francis Hirsch for a private Note concerning a simple calculation and Professor Zeev Schuss for his valuable advice in order to improve some important aspects on the text.

This work would have not been accomplished without the support of the Human Frontier Science Program and of the Empeirikeion Foundation from

Greece.

Additionally, I feel lucky that I had the chance to participate in Marc Yor's WIP, a lab meeting that many well known people have attended. I am indebted especially to the Professors Cathérine Donati-Martin and Frédérique Petit for their constant presence in the WIP and their very useful remarks. The interchanges with the other PhD students participating in the WIP during these years and the opportunity to present and discuss my research and my progression with them, were indispensable for continuing-and finally achieving- my doctorate. More specifically, I would like to thank Joseph, Nizar, Paul, Fernando, Cengbo, Joachim, David, Xinxin and Julia in particular.

In addition, I address my thanks to the members of the lab-meeting of David Holcman's group "Modeling of Cellular Physiology" at the Ecole Normale Supérieure of Paris. I also presented and discussed my progress and my difficulties in order to proceed many times. I especially thank Juergen and Nathanaël for their help with the Simulations and Krasi, Emmanuel, Thibault, Godefroy, Carlo, Khahn, Jérémie, Assaf, Qiang, Ofir and Claire for their enjoyable company over the past few years.

I should also thank all my colleagues from my office at Pierre et Marie Curie University (at Chevaleret in the beginning and at Jussieu since the last summer). The relaxing times and the fun that we had in the breaks encouraged me to proceed with my research. So, I thank everyone from the Laboratory of Probability and Random Models, and especially Abass, Alexander, Camille, Cécile, Eric, Emmanuel, François, Guanying, Jean-Paul, Kilian, Mathieu, Nathalie, Nikos, Nouffel, Pascal, Raoul, Sophie D., Sophie L., Sylvain, Thanh Mai, Vincent and any others that I may have forgotten...

Moreover, I should mention my colleagues from the office at the University of Nanterre, where I worked during the last 2 years as an ATER and the whole team of MODAL'X who have welcomed me cordially. Cyrille, Ines, Malika, Olivier, Philippe, Sébastien, Rudiger, Rym, Valérie, thank you for the nice time that we had there.

I also owe many thanks to the members of the administration Jean-Christophe Cazenave, Valérie Juvé, Corentin Lacombe, Philippe Macé, Isabelle Mariage, Maria Pochot, Jacques Portès, Josette Saman and Jean-François Venuti for their help during all these years in order to accomplish successfully this task.

I should not forget to thank my Professor from the National Technical University of Athens in Greece, Ioannis Spiliotis, who was the first who introduced me in the theory of Advanced Probability and Stochastic Calculus. Working with him, provided me with the desire to continue studying, and later on researching, in this direction.

In addition, I feel very grateful for the friendships that I made here in Paris. Especially, I would like to thank Katerina for her help with the figures in this PhD thesis.

Finally, I would like to thank my parents and my sister Nicole for their support, Eytyxia who "suffered" a long and difficult period, my friends and my relatives. I am certain that, without them, I would not have achieved finishing my PhD.

Σας ευχαριστώ πολύ !!!!

Η μόνη χαμένη μάχη είναι αυτή που δε δόθηκε ποτέ.

Abbreviations

$a_s(x) \equiv \arg\sinh(x) \equiv \log(x + \sqrt{x^2+1}),\ x \in \mathbb{R}$

$a_c(y) \equiv \arg\cosh(y) \equiv \log(y + \sqrt{y^2-1}),\ y \geq 1$

a.s.: almost sure

B.C.: boundary condition

BDG: Burkholder-Davis-Gundy inequality

BM: Brownian motion

DDS: Dambis-Dubins-Schwarz

GGC: Generalized Gamma Convolution

GOUP: generalized Ornstein-Uhlenbeck process

I.C.: initial condition

i.i.d.: independent and identically distributed

$\overset{L^2}{\approx}$: approximation in mean square (in the sense of the L^2-norm)

$\xrightarrow{(law)}$: convergence in law

$\overset{(law)}{=}$: equality in law (or equality in distribution)

LHS: left hand side

MRT: Mean Rotation Time

MFPT: Mean First Passage Time

MMRT: Minimum Mean Rotation Time

OU: Ornstein-Uhlenbeck

OUP: Ornstein-Uhlenbeck process

$\xrightarrow{(P)}$: convergence in Probability

$P - \lim$: Probability limit

RHS: right hand side

TRM: temps de la premiére rotation en moyenne

Contents

List of Figures

Publications

1. S. Vakeroudis (2010). On hitting times of the winding processes of planar Brownian motion and of Ornstein-Uhlenbeck processes, via Bougerol's identity. Submitted to SIAM TVP Journal.

2. S. Vakeroudis, M. Yor and D. Holcman (2011). The Mean First Rotation Time of a planar polymer. To appear in Journal of Statistical Physics.

3. S. Vakeroudis (2011). On some infinite divisibility properties related with Bougerol's identity. In Preparation.

Chapter 1

Introduction

1.1 Some history on Brownian motion and new results

The conformal invariance of planar Brownian motion[†] has deep consequences as to the structure of its trajectories [LeG90]. What distinguishes dimension two from higher dimensions is that the planar BM is *neighborhood recurrent*, which means that for any open domain S in the plane, in large time scales ($t \to \infty$), the time that it spends in S tends to ∞. Moreover, what distinguishes dimension two from dimension one is that the planar BM Z does not hit points almost surely (a.s.), for each point z in the plane (different from the initial position Z_0),

$$P\left(Z_t = z, t > 0\right) = 0.$$

This means that if $Z_0 \neq 0$, Z does not visit a.s. the point 0; however, it keeps winding around 0 infinitely often. Thus, we can define its continuous winding process $\theta_t = \mathrm{Im}(\int_0^t \frac{dZ_s}{Z_s}), t \geq 0$. In particular, a number of articles have been devoted to the study of this continuous winding process $(\theta_t, t \geq 0)$: [Spi58, Wil74, Dur82, MeY82, PiY86, LeGY87, BeW94, Yor97, PaY00, BPY03]. More precisely, Spitzer in (1958) [Spi58] showed that θ_t normalized by $\frac{\log t}{2}$, converges in law, as $t \to \infty$, to a standard Cauchy variable C_1, which is

[†]When we simply write: Brownian motion, we always mean real-valued Brownian motion, starting from 0. For 2-dimensional Brownian motion, we indicate planar or complex BM.

called *Spitzer's law/theorem*:

$$\frac{2}{\log t}\,\theta_t \xrightarrow[t\to\infty]{(law)} C_1 \ ,$$

with:

$$P\left(C_1 \in dx\right)/dx = \frac{1}{\pi\left(1+x^2\right)}, \qquad -\infty < x < \infty.$$

Williams [Wil74], gave a non-computational proof of Spitzer's law, based on his *"pinching method"*. Following this method, in order to prove Spitzer's law, one needs only prove the - by far - easier result that

$$\frac{1}{\log r}\,\theta_{T(r)} \xrightarrow[t\to\infty]{(law)} C_1 \ ,$$

where $T(r) \equiv \inf\{t \geq 0 : |Z_t| > r\}$ [Wil74, MeY82]. Another proof of Spitzer's theorem was given by Durrett [Dur82]. In this article, the author proposes a proof based on Lévy's result on the conformal invariance of BM. The study of the asymptotic behavior of planar BM continued with the new idea of decomposing the continuous winding process of planar BM into a process of big windings and a process of small windings [MeY82]. Following [PiY86], let D_+ (the big domain) and D_- (the small domain) be the open sets inside and outside the unit circle. The sign + and - indicates big and small respectively. We define:

$$\theta_\pm(t) = \int_0^t 1(Z(s) \in D_\pm)\, d\theta_s,$$

where $1(A)$ stands for the indicator of A.

The process θ_+ is the process of big windings and θ_- is the process of small windings. Because the Lebesgue measure of the time spent by Z on the unit circle is a.s. 0, there is the identity:

$$\theta = \theta_+ + \theta_-.$$

This decomposition describes the winding process θ as alternating between long stretches of time, when Z is far away from the origin in D_+ and θ changes very slowly (but significantly) with respect to θ_+, and small stretches of time, when Z is in D_- approaching 0 and θ changes very rapidly with respect to θ_-.

It turns out that only the very big windings and very small windings count for the asymptotic behavior (as $t \to \infty$) of the total winding and moreover, the windings for a very large class of two-dimensional random walks, behave rather more like θ_+ than θ [BeW94, Shi98].

The study of two dimensional random walks and of planar BM, based on Spitzer's law, has been continued by several researchers [Spi64, Wie84, Bel86, Bel89, BeF91]. Another idea used by several authors [RuH87, RuH88, CDM93, Sal94] etc., was to consider the -so called- self-avoiding random walk (a random walk that does not visit the points that it has already visited, thus an "excluded" volume is considered in a neighborhood around the origin). Windings of more general planar Gaussian processes remain a subject of research [LDEH09].

Most of the studies made up to now, concern the asymptotic behavior of the continuous winding process of planar BM. However, there is not so much literature concerning the first time of a winding of given amplitude (an aspect which has many applications, for instance in Biology and more specifically in polymers). Based on this kind of questions, this PhD thesis is devoted to the study of several such hitting times. Finally, we give some applications concerning planar polymers.

In Biology, the idea of modeling an ion, a molecule or a protein moving into a compartment or a cell as a Brownian particle moving into a bounded domain has been widely used during the last years. The *narrow escape problem* (also known as *"small hole theory"*) introduced by Schuss-Singer-Holcman [SSH07] gave a new direction of research in Biology and motivated also the use of Brownian motion in the approach that we propose. More precisely, the authors study a Brownian particle moving into a bounded domain searching a small hole (in dimensions two and three) and they give some asymptotic formulas for the mean escape time (Mean First Passage Time - MFPT) which represents the mean time it takes for a molecule to hit a binding site.

In this research, we take up again the study of the first hitting times:

$$T^{\theta}_{-d,c} \equiv \inf\{t : \theta_t \notin (-d, c)\}, \ (c, d > 0),$$

this time in relation with Bougerol's well-known identity [Bou83, ADY97, Yor01]: for fixed $u > 0$,

$$\sinh(\beta_u) \stackrel{(law)}{=} \hat{\beta}_{\left(\int_0^u ds \exp(2\beta_s)\right)} \ ,$$

where $(\hat{\beta}_t, t \geq 0)$ is a BM, independent of β^*. In particular, it turns out that: for fixed $c > 0$:

$$\theta_{T_c^{\hat{\beta}}} \stackrel{(law)}{=} C_{a(c)}, \qquad (\star)$$

where $\hat{\beta}$ is a BM independent of $(\theta_u, u \geq 0)$, $T_c^{\hat{\beta}} = \inf\{t : \hat{\beta}_t = c\}$, $(C_t, t \geq 0)$ is a standard Cauchy process and $a(c) = \arg\sinh(c) \equiv \log\left(c + \sqrt{1+c^2}\right)$, $c \in \mathbb{R}$. From this identity in law, we obtain a "pseudo"-Laplace transform, an idea which was initially used in [Yor01].

The identity $(\star)$ yields yet another proof of the celebrated Spitzer theorem:

$$\frac{2}{\log t}\,\theta_t \xrightarrow[t\to\infty]{(law)} C_1\ ,$$

with the help of Williams' "pinching method" [Wil74, MeY82].

Moreover, we study the distributions of $T^{\theta}_{-\infty,c}$ and $T^{\theta}_{-c,c}$. In particular, we give an explicit formula for the density function of $T^{\theta}_{-c,c}$ and for the first moment of $\ln\left(T^{\theta}_{-c,c}\right)$.

Going a little further, we develop similar results when planar Brownian motion is replaced by a complex valued Ornstein-Uhlenbeck process. We note that in [BeW94] there are already some discussions of windings for planar Brownian motion using arguments related to Ornstein-Uhlenbeck processes. Firstly, we obtain some analogue of $(\star)$ when $T_c^{\hat{\beta}}$ is replaced by $T_c^{(\lambda)} = T_c^{\theta} = \inf\{t : |\theta_t^Z| = c\}$, the corresponding time for an Ornstein-Uhlenbeck process Z with parameter λ. Secondly, we exhibit the distribution of $T_c^{(\lambda)}$. More specifically, we derive the asymptotics of $E\left[T_c^{(\lambda)}\right]$ for λ large and for λ small.

Finally, always in relation to Bougerol's identity, we show that $1/A_t(\beta)$ (where $A_t(\beta) = \int_0^t ds \exp(2\beta_s)$ is the time scale of the BM in Bougerol's identity), considered after a suitable measure change from Wiener measure, $P_t = \left(\frac{t}{A_t(\beta)}\right)^{1/2} \cdot W$, where W is Wiener measure, is infinitely divisible. More specifically, we show that $1/A_t(\beta)$ is decomposed as the sum of N^2 (with N denoting a standard reduced Gaussian variable), and a subordinator which is an increasing Lévy process (i.e. a process $Y = (Y_t, t \geq 0)$ with $Y_0 = 0$ a.s., with stationary and independent increments and which is almost surely right continuous with left limits).

*Note that, clearly, the two processes are not identical in law, as the RHS is a martingale, whereas the LHS is not.

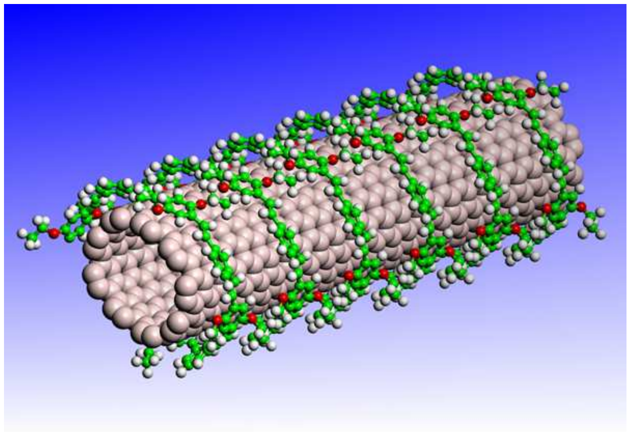


Figure 1.1: **Pinned polymer in 3D.** We are interested in the first time the polymer makes one single loop around the filament.

1.2 Elements of Biology and connection with Stochastic Processes

Because a polymer can change its motion when it loops around a filament (from 3 dimensions to 1 dimension), we aim to characterize the mean time for a rotation to be completed for the first time and in particular, how it depends on various parameters such as the polymer length, the diffusion coefficient or the distance from a pinned polymer end. We named this time, the mean rotation time (MRT) and we present here some asymptotic computations when the rotation occurs around a fixed filament. In dimension two, we restrict our analysis to the planar polymer rotation around a point. The MRT provides a quantification for the transition from a free three or two dimensional Brownian motion to a restricted one dimensional motion, after a rotation is completed (see Figure 1.1). In Figure 1.2 we see a simplified picture of a polymer making a loop around a filament. The 3D to 1D transition does not require any physical binding or molecular interaction: the polymer is simply trapped by this winding loop. This idea can be used to characterize the transition motion of a DNA molecule or a plasmid moving inside the cell cytoplasm. To characterize this phenomena, we propose to use a simplified model of a polymer made by a collection of two dimensional connected rods with random angles (Figure 1.3). We shall fix one extreme part of the polymer and we propose to estimate the mean rotation time (MRT) of the rest

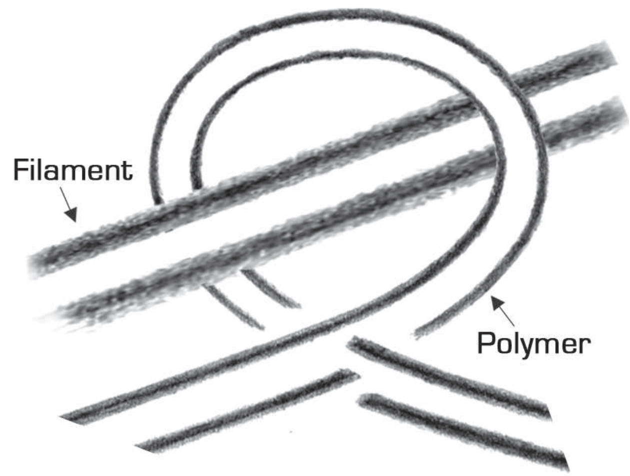


Figure 1.2: **A single loop of a pinned polymer in 3D.** The polymer makes one loop around the filament.

of the polymer around the origin. We shall examine how the MRT depends on various parameters such as the diffusion constant, the number of rods or their common length. Related to this issue, another characterization is obtained by the mean time for a polymer to loop [WiF74, PZS96]. Moreover, some studies concerning the winding of directed polymers around each other or around a rod have already been done [DrK96].

Various models are available to study polymers: the Rouse model consists of a collection of beads connected by springs, while more sophisticated models account for bending, torsion and specific mechanical properties [Rou53, SSS80, DoE94]. We shall consider here a very crude approximation where a planar polymer is modeled as a collection of n rigid rods, with equal fixed lengths l_0 and we denote their extremities by $(X_0, X_1, X_2, \ldots, X_n)$ (Figure 1.3) in a framework with an origin $\mathbf{0}$. We shall fix one of the polymer ends $X_0 = (L, 0)$ (where $L > 0$) on the x-axis. The dynamics of the i-th rod is characterized by its angle $\theta_i(t)$ with respect to the x-axis. The overall polymer dynamics is thus characterized by the angles $(\theta_1(t), \theta_2(t), \ldots, \theta_n(t), t \geq 0)$. Due to the thermal collisions in the medium,

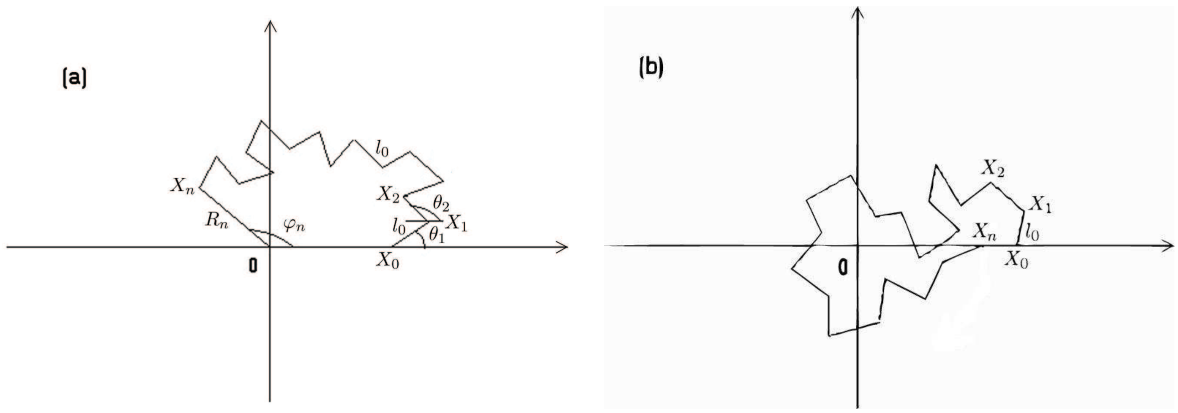


Figure 1.3: **Schematic representation of a planar polymer winding around the origin.** (a)A random configuration, (b)MRT when the n-th bead reaches 2π.

each angle follows a Brownian motion. Thus,

$$(\theta_i(t), i \leq n) \stackrel{(law)}{=} \sqrt{2D}\,(B_i(t), i \leq n) \Leftrightarrow \begin{cases} d\theta_1(t) = \sqrt{2D}\,dB_1(t) \\ \\ d\theta_2(t) = \sqrt{2D}\,dB_2(t) \\ \\ \vdots \\ d\theta_n(t) = \sqrt{2D}\,dB_n(t), \end{cases}$$

where D is the rotational diffusion constant and $(B_1(t), \ldots, B_n(t), t \geq 0)$ is an n-dimensional Brownian motion (BM). The position of each rod can now be obtained as:

$$\begin{cases} X_1(t) = L + l_0 e^{i\theta_1(t)} \\ \\ X_2(t) = X_1(t) + l_0 e^{i\theta_2(t)} \\ \\ \vdots \\ X_n(t) = X_{n-1}(t) + l_0 e^{i\theta_n(t)}. \end{cases} \tag{1.1}$$

In particular, the moving end is given by:

$$\begin{aligned} X_n(t) &= L + l_0 \left(e^{i\theta_1(t)} + \ldots + e^{i\theta_n(t)}\right) = L + l_0 \sum_{k=1}^{n} e^{i\theta_k(t)} \\ &= L + l_0 \sum_{k=1}^{n} e^{i\sqrt{2D}B_k(t)}, \end{aligned} \tag{1.2}$$

which can be written as:

$$X_n(t) = R_n(t)e^{i\varphi_n(t)}, \tag{1.3}$$

and thus $\varphi_n(t)$ accounts for the rotation of the polymer with respect to the origin $\mathbf{0}$ and R_n is the distance to the origin.

In order to compute the MRT, we shall study a sum of exponentials of Brownian motions, a topic which often leads to surprising computations [Yor01]. Moreover, the imaginary exponentials of Brownian motion, remains a subject of research [GDL11].
First, we scale the space and time variables as follows:

$$\tilde{l} = \frac{L}{l_0} \quad \text{and} \quad \tilde{t} = \frac{t}{2D}. \tag{1.4}$$

Equation (1.2) becomes:

$$X_n(t) = \tilde{l} + \sum_{k=1}^{n} e^{i\tilde{B}_k(t)}, \tag{1.5}$$

where $\left(\tilde{B}_1(t), \ldots, \tilde{B}_n(t), t \geq 0\right)$ is an n-dimensional Brownian motion (BM), and for $k = 1, \ldots, n$, using the scaling property of Brownian motion, we have:

$$\tilde{B}_k(t) \equiv \frac{1}{\sqrt{2D}} B_k(t) \overset{(law)}{=} B_k\left(\frac{t}{2D}\right) = B_k(\tilde{t}).$$

Before describing our approach, we first discuss the mean initial configuration of the polymer. It is given by:

$$c_n = E\left(\sum_{k=1}^{n} e^{i\theta_k(0)}\right), \tag{1.6}$$

where the initial angles $\theta_k(0)$ are such that the polymer has not already made a loop. In order to study the MRT, it turns out that we have to address the same questions for an Ornstein-Uhlenbeck process and for a planar BM (which is the subject of the first part of this PhD thesis).

The following text consists of three more chapters and an Appendix, as follows:

- The second chapter is devoted to Stochastic processes and more precisely to the study of planar Brownian motion and of complex valued Ornstein-Uhlenbeck processes. It reproduces more or less the following article: **"On hitting times of the winding processes of planar Brownian motion and of Ornstein-Uhlenbeck processes, via Bougerol's identity", S. Vakeroudis (2010), Submitted to SIAM TVP Journal.** Here, we develop some identities in law in terms of planar complex valued Ornstein-Uhlenbeck processes ($Z_t = X_t + iY_t, t \geq 0$) including planar Brownian motion, which are equivalent to the well known Bougerol identity for linear Brownian motion ($\beta_t, t \geq 0$): for any fixed $u > 0$,

$$\sinh(\beta_u) \overset{(law)}{=} \hat{\beta}_{(\int_0^u ds\, \exp(2\beta_s))},$$

  with ($\hat{\beta}_t, t \geq 0$) a Brownian motion, independent of β.
  These identities in law for 2-dimensional processes allow us to study the distribution of hitting times $T_c^\theta \equiv \inf\{t : \theta_t = c\}$, $(c > 0)$, $T_{-d,c}^\theta \equiv \inf\{t : \theta_t \notin (-d, c)\}$, $(c, d > 0)$ and more specifically of $T_{-c,c}^\theta \equiv \inf\{t : \theta_t \notin (-c, c)\}$, $(c > 0)$ of the continuous winding processes $\theta_t = \mathrm{Im}(\int_0^t \frac{dZ_s}{Z_s}), t \geq 0$ associated to complex Ornstein-Uhlenbeck processes ($Z_s, s \geq 0$).

  Moreover, we show that the inverse of the new time scale of the Brownian motion in Bougerol's identity $1/A_u(\beta)$ (with $A_u(\beta) = \int_0^t ds \exp(2\beta_s)$), considered after a suitable measure change from Wiener measure, is infinitely divisible. This is revisited in the article **"On some infinite divisibility properties related with Bougerol's identity", S. Vakeroudis (2011), In Preparation**.

- The third chapter is devoted to the biological problem of the rotation (winding) of a planar polymer. This chapter reproduces the following article: **"The Mean First Rotation Time of a planar polymer", S. Vakeroudis, M. Yor and D. Holcman (2011), To appear in Journal of Statistical Physics.** First, we explain the way that we model its 2-dimensional movement. We prove a Central Limit Theorem and we use the results so as to prove that finally, its movement may be approximated by an Ornstein-Uhlenbeck process with nonlinear time function (drift). Thus, we use the results of the second chapter and we derive an asymptotic formula for the MRT $E\left[\tau_n\right]$ when the polymer is made of n rods of equal lengths l_0 with the first end fixed at a distance L from the origin and the Brownian motion is characterized by the diffusion constant D the rotational diffusion constant. The MRT depends on $\sqrt{n}$ and, logarithmically on the mean initial configuration and for $n\, l_0 >> L$ and $n \geq 3$, the leading order term is given by:

  1. for a general initial configuration:
  $$E\left[\tau_n\right] \approx \frac{\sqrt{n}}{8D}\left[2\ln\left(\frac{c_n}{\sqrt{n}}\right) + 0.08\frac{n}{c_n^2} + Q\right],$$
  where $c_n \equiv E\left(\sum_{k=1}^{n} e^{\frac{i}{\sqrt{2D}}\theta_k(0)}\right)$, $(\theta_k(0), 1 \leq k \leq n)$ is the sequence of the initial angles and $Q \approx 9.54$,
  2. for a stretched initial configuration:
  $$E\left[\tau_n\right] \approx \frac{\sqrt{n}}{8D}\,\left(\ln(n) + Q\right),$$
  3. for an average over uniformly distributed initial angles:
  $$E\left[\tau_n\right] \approx \frac{\sqrt{n}}{8D}\,\tilde{Q},$$
  where $\tilde{Q} \approx 9.62$.

  Finally, we confirm our analytical results with Brownian simulations.

- The fourth chapter gives some perspectives for further research and extensions that we may obtain in both areas of Mathematics and Biology.

- Finally, in the Appendix, we give a table with Bougerol's identity and other equivalent expressions, we show (by explicit calculations) that $\log(\sqrt{x}+\sqrt{1+x})$ and $(\log(\sqrt{x}+\sqrt{1+x}))^2$ are Bernstein functions, we give the proof of Theorem 3.2.1 and, by supposing that θ_1 is a reflected Brownian motion in $[0, 2\pi]$, we calculate and use its probability density function to provide a more accurate estimation of the MRT.

We note that the chapters of this PhD thesis are more or less independent of each other.

Chapter 2

On hitting times of the winding processes of planar Brownian motion and of Ornstein-Uhlenbeck processes, via Bougerol's identity.

2.1 The Brownian motion case

2.1.1 A reminder on planar Brownian motion

Let $(Z_t = X_t + iY_t, t \geq 0)$ denote a standard planar Brownian motion, starting from $x_0 + i0, x_0 > 0$, where $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ are two independent linear Brownian motions, starting respectively from x_0 and 0.
As is well known [ItMK65], since $x_0 \neq 0$, $(Z_t, t \geq 0)$ does not visit a.s. the point 0 but keeps winding around 0 infinitely often. In particular, the continuous winding process $\theta_t = \mathrm{Im}(\int_0^t \frac{dZ_s}{Z_s}), t \geq 0$ is well defined.
Furthermore, there is the skew product representation:

$$\log |Z_t| + i\theta_t \equiv \int_0^t \frac{dZ_s}{Z_s} = (\beta_u + i\gamma_u) \Big|_{u=H_t=\int_0^t \frac{ds}{|Z_s|^2}} , \tag{2.1}$$

where $(\beta_u + i\gamma_u, u \geq 0)$ is another planar Brownian motion starting from $\log x_0 + i0$ (for further study of the Bessel clock H, see [Yor80]).

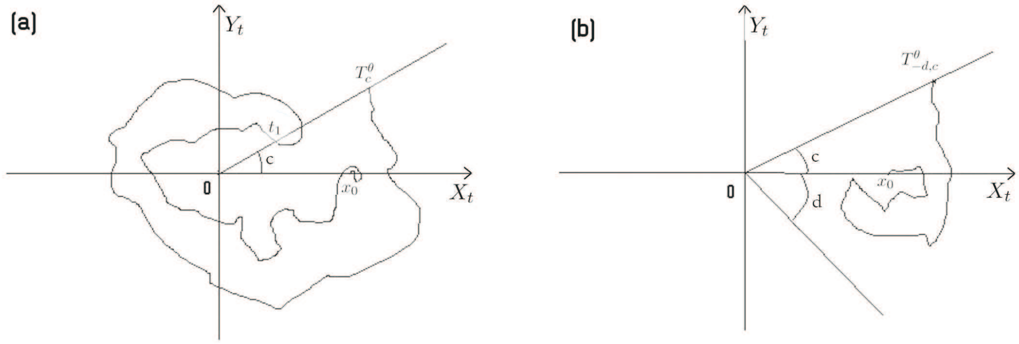


Figure 2.1: **Exit times for a planar BM.** This Figure presents the Exit times (a) T_c^θ (t_1 doesn't count because the angle is negative) and (b) $T_{-d,c}^\theta$ for a planar BM starting from $x_0 + i0$.

Rewriting (2.1) as:

$$\log |Z_t| = \beta_{H_t}; \quad \theta_t = \gamma_{H_t}, \tag{2.2}$$

we easily obtain that the two σ-fields $\sigma\{|Z_t|, t \geq 0\}$ and $\sigma\{\beta_u, u \geq 0\}$ are identical, whereas $(\gamma_u, u \geq 0)$ is independent from $(|Z_t|, t \geq 0)$.

A number of studies of the properties of the first hitting time (see Figure 2.1(b))

$$T_{-d,c}^\theta \equiv \inf\{t : \theta_t \notin (-d, c)\}, \quad (c, d > 0),$$

have been developed [Spi58].
In particular, it is well known [Spi58, Bur77] and [ReY99] (Ex. 2.21/page 196), that:

$$E\left[(T_{-d,c}^\theta)^p\right] < \infty \quad \text{if and only if} \quad p < \frac{\pi}{2(c+d)}. \tag{2.3}$$

Moreover, once again,Spitzer's asymptotic theorem [Spi58] states that:

$$\frac{2\theta_t}{\log t} \underset{t\to\infty}{\overset{(law)}{\longrightarrow}} C_1 \overset{(law)}{=} \gamma_{T_1^\beta}, \tag{2.4}$$

where C_1 is a standard Cauchy variable.

2.1.2 On the Laplace transform of the distribution of the hitting time $T_c^\theta \equiv T_{-\infty,c}^\theta$

Now, we use the representation (2.2) to access the distribution of T_c^θ (see Figure 2.1(a)). We define $T_c^\gamma \equiv \inf\{t : \gamma_t \notin (-\infty, c)\}$ the hitting time associated to the Brownian motion $(\gamma_t, t \geq 0)$. Note that:
from (2.2), $H_{T_c^\theta} = T_c^\gamma$, hence $T_c^\theta = H_u^{-1}\Big|_{u=T_c^\gamma}$, where

$$H_u^{-1} \equiv \inf\{t : H_t > u\} = \int_0^u ds \exp(2\beta_s) := A_u(\beta). \tag{2.5}$$

From now on, we shall use A instead of $A(\beta)$.Thus, we have obtained:

$$T_c^\theta = A_{T_c^\gamma}, \tag{2.6}$$

where $(A_u, u \geq 0)$ and T_c^γ are independent, since β and γ are independent. We can write: $\beta_s = (\log x_0) + \beta_s^{(0)}$, with $(\beta_s^{(0)}, s \geq 0)$ a standard one-dimensional Brownian motion starting from 0. Then, we deduce from (2.6) that:

$$T_c^\theta = x_0^2 \left(\int_0^{T_c^\gamma} ds \exp(2\beta_s^{(0)}) \right). \tag{2.7}$$

From now on, for simplicity, we shall take $x_0 = 1$, but this is really no restriction, as the dependency in x_0, which is exhibited in (2.7), is very simple.
We shall also make use of Bougerol's identity [Bou83, ADY97] and [Yor01] (p. 200), which is very useful to study the distribution of A_u (e.g. [MaY98, MaY05]). For any fixed $u > 0$,

$$\sinh(\beta_u) \overset{(law)}{=} \hat{\beta}_{A_u} = \hat{\beta}_{(\int_0^u ds \exp(2\beta_s))}, \tag{2.8}$$

where on the right hand side, $(\hat{\beta}_t, t \geq 0)$ is a Brownian motion, independent of $A_u \equiv \int_0^u ds \exp(2\beta_s)$.
Thus, from (2.8) and (2.6), and as is well known [ReY99], the law of $\beta_{T_c^\gamma}$ is the Cauchy law with parameter c, i.e., with density:

$$h_c(y) = \frac{c}{\pi(c^2 + y^2)},$$

we deduce that:

Proposition 2.1.1 *For fixed $c > 0$, there is the following identity in law:*

$$\sinh(C_c) \overset{(law)}{=} \hat{\beta}_{(T_c^\theta)}, \tag{2.9}$$

where, on the left hand side, $(C_c, c \geq 0)$ denotes a standard Cauchy process and on the right hand side, $(\hat{\beta}_u, u \geq 0)$ is a one-dimensional BM, independent from T_c^θ.

We may now identify the densities of the variables found on both sides of (2.9), i.e.:
on the left hand side: $\frac{1}{\sqrt{1+x^2}} h_c(\arg\sinh x) = \frac{1}{\sqrt{1+x^2}} h_c(a(x))$;
on the right hand side: $E\left[\frac{1}{\sqrt{2\pi T_c^\theta}} \exp\left(-\frac{x^2}{2T_c^\theta}\right)\right]$,
where $a(x) = \arg\sinh(x)$.
Thus, we have obtained the following:

Proposition 2.1.2 *The distribution of T_c^θ is characterized by:*

$$E\left[\frac{1}{\sqrt{2\pi T_c^\theta}} \exp\left(-\frac{x}{2T_c^\theta}\right)\right] = \frac{1}{\sqrt{1+x}} \frac{c}{\pi(c^2 + \log^2(\sqrt{x} + \sqrt{1+x}))}, \quad x \geq 0. \tag{2.10}$$

The proof of Proposition 2.1.2 follows from: $a(y) = \arg\sinh(y) \equiv \log(y + \sqrt{1+y^2})$ and by making the change of variable $y^2 = x$. Let us now define the probability:

$$Q_c = \sqrt{\frac{\pi}{2T_c^\theta}}\, c \cdot P\ .$$

The fact that Q_c is a probability follows from (2.10) by taking $x = 0$. Thus we obtain that $c\, E[\sqrt{\pi/2T_c^\theta}] = 1$, and we may write:

$$E_{Q_c}\left[\exp\left(-\frac{x}{2T_c^\theta}\right)\right] = \frac{1}{\sqrt{1+x}} \frac{1}{1 + \frac{1}{c^2}\log^2(\sqrt{x} + \sqrt{1+x})}, \quad \forall x \geq 0, \tag{2.11}$$

which yields the Laplace transform of $1/T_c^\theta$ under Q_c.
Let us now take a look at what happens if we make $c \to \infty$. If we denote by $T_1^\beta \equiv \inf\{t : \beta_t = 1\}$ the first hitting time of level 1 for a standard BM

β and by N a standard Gaussian variable $\mathcal{N}(0,1)$, from equation (2.11), we obtain:

$$\lim_{c\to\infty} E_{Q_c}\left[e^{-x/2T_c^\theta}\right] = E\left(e^{-xN^2/2}\right) = E\left(e^{-x/2T_1^\beta}\right), \tag{2.12}$$

which means that : $T_c^\theta \underset{c\to\infty}{\overset{(law)}{\longrightarrow}} T_1^\beta$. (At this point, one may wonder whether there is some kind of convergence in law involving $(\theta_u, u \geq 0)$, under Q_c, as $c \to \infty$, but, we shall not touch this point).

From Proposition 2.1.2 we deduce the following:

Corollary 2.1.3 *Let $\varphi(x)$ denote the Laplace transform of Proposition 2.1.2 i.e. the expression on the right hand side of (2.10), which is the Laplace transform of $1/2T_c^\theta$ under Q_c. Then, the Laplace transform of $1/2T_c^\theta$ under P is:*

$$E\left[\exp\left(-\frac{x}{2T_c^\theta}\right)\right] = \int_x^\infty \frac{dw}{\sqrt{w-x}}\,\varphi(w). \tag{2.13}$$

Proof of Corollary 2.1.3 From Fubini's theorem, we deduce from (2.11) that:

$$\begin{aligned}
E\left[\exp\left(-\frac{x}{2T_c^\theta}\right)\right] &= \int_0^\infty \frac{dy}{\sqrt{y}}\, E\left[\frac{1}{\sqrt{2\pi T_c^\theta}}\exp\left(-\frac{x+y}{2T_c^\theta}\right)\right] \\
&= \int_0^\infty \frac{dy}{\sqrt{y}}\,\varphi(x+y) \\
&\overset{y=xt}{=} \sqrt{x}\int_0^\infty \frac{dt}{\sqrt{t}}\,\varphi(x(1+t)) \\
&\overset{v=1+t}{=} \sqrt{x}\int_1^\infty \frac{dv}{\sqrt{v-1}}\,\varphi(xv) \\
&\overset{w=xv}{=} \int_x^\infty \frac{dw}{\sqrt{w-x}}\,\varphi(w),
\end{aligned}$$

which is formula (2.13).

$\square$

2.1.3 Some related identities in law

This subsection is strongly related to [DuY11].
A slightly different look at the combination of Bougerol's identity (2.8) and the skew-product representation (2.1) leads to the following striking identities in law:

Proposition 2.1.4 *Let $(\delta_u, u \geq 0)$ be a 1-dimensional Brownian motion independent of the planar Brownian motion $(Z_u, u \geq 0)$, starting from 1. Then, for any $b \geq 0$, the following identities in law hold:*

$$\text{(i) } H_{T_b^\delta} \stackrel{(law)}{=} T^\beta_{a(b)} \qquad \text{(ii) } \theta_{T_b^\delta} \stackrel{(law)}{=} C_{a(b)} \qquad \text{(iii) } \bar{\theta}_{T_b^\delta} \stackrel{(law)}{=} |C_{a(b)}|,$$

where C_A is a Cauchy variable with parameter A and $\bar{\theta}_u = \sup_{s\leq u} \theta_s$.

Remark 2.1.5 *All the expressions which are equivalent to Bougerol's identity and are used and proved in this PhD thesis are given in a table in the Appendix A.1.*

Proof of Proposition 2.1.4 From the symmetry principle ([And87] for the original Note and [Gal08] for a detailed discussion), Bougerol's identity may be equivalently stated as:

$$\sinh(\bar{\beta}_u) \stackrel{(law)}{=} \bar{\delta}_{A_u(\beta)}. \tag{2.14}$$

Consequently, the laws of the first hitting times of a fixed level b by the processes on each side of (2.14) are identical, that is:

$$T^\beta_{a(b)} \stackrel{(law)}{=} H_{T_b^\delta},$$

which is (i).
(ii) follows from (i) since:

$$\theta_u \stackrel{(law)}{=} \gamma_{H_u},$$

with $(\gamma_s, s \geq 0)$ a Brownian motion independent of $(H_u, u \geq 0)$ and $(C_u, u \geq 0)$ may be represented as $(\gamma_{T_u^\beta}, u \geq 0)$.
(iii) follows from (ii), again with the help of the symmetry principle.

$\square$

Remark 2.1.6 *(It may be skipped by the reader)*
Proposition 2.1.2 may be derived from (iii) in Proposition 2.1.4. Indeed, for $c > 0$, we have: on the LHS of (iii) (let $N \sim \mathcal{N}(0,1)$ with N independent from T_c^θ):

$$\begin{aligned} P\left(\bar{\theta}_{T_b^\delta} < c\right) &= P\left(T_b^\delta < T_c^\theta\right) = P\left(b < \bar{\delta}_{T_c^\theta}\right) \\ &= P\left(b < \sqrt{T_c^\theta}|N|\right) \\ &= P\left(\frac{b}{\sqrt{T_c^\theta}} < |N|\right) \\ &= \sqrt{\frac{2}{\pi}} E\left[\int_{b/\sqrt{T_c^\theta}}^\infty dy\, e^{-y^2/2}\right], \end{aligned} \tag{2.15}$$

and on the RHS of (iii):

$$P\left(|C_{a(b)}| < c\right) = 2\int_0^c \frac{a(b)\, dy}{\pi(a^2(b) + y^2)} \overset{y=a(b)h}{=} \frac{2}{\pi}\int_0^{c/a(b)} \frac{dh}{1+h^2}. \tag{2.16}$$

Taking derivatives in (2.15) and (2.16) with respect to b and changing the variables $b = \sqrt{x}$, we obtain Proposition 2.1.2.

2.1.4 Recovering Spitzer's theorem

The identity (ii) in Proposition 2.1.4 is reminiscent of Williams' remark [Wil74, MeY82], that:

$$H_{T_r^R} \overset{(law)}{=} T_{\log r}^\delta, \tag{2.17}$$

where here R starts from 1 and δ starts from 0 (in fact, this is a consequence of (2.2)). For a number of variants of (2.17) [Yor85, MaY08]. This was D. Williams' starting point for a non-computational proof of Spitzer's result (2.4). We note that in (ii), T_b^δ is independent of the process $(\theta_u, u \geq 0)$ while in (2.17) T_r^R depends on $(\theta_u, u \geq 0)$. Actually, we can mimic Williams' "pinching method" to derive Spitzer's theorem (2.4) from (ii) in Proposition 2.1.4.

Proposition 2.1.7 ***(A new proof of Spitzer's theorem)***
As $t \to \infty$, $\theta_{T^{\delta}_{\sqrt{t}}} - \theta_t$ converges in law, which implies that:

$$\frac{1}{\log t}\left(\theta_{T^{\delta}_{\sqrt{t}}} - \theta_t\right) \xrightarrow[t\to\infty]{(P)} 0, \tag{2.18}$$

which, in turn, implies Spitzer's theorem (see formula (2.4)):

$$\frac{2}{\log t}\,\theta_t \xrightarrow[t\to\infty]{(law)} C_1.$$

Proof of Proposition 2.1.7 From equation (ii) of Proposition 2.1.4 we note:

$$\frac{1}{\log b}\,\theta_{T^{\delta}_{b}} \overset{(law)}{=} \frac{C_{a(b)}}{\log b} \xrightarrow[b\to\infty]{(law)} C_1.$$

So, taking $b = \sqrt{t}$ we have:

$$\frac{2}{\log t}\,\theta_{T^{\delta}_{\sqrt{t}}} \xrightarrow[t\to\infty]{(law)} C_1.$$

On the other hand, following Williams' "pinching method" idea, we note that:

$$\frac{1}{\log t}\left(\theta_{T^{\delta}_{\sqrt{t}}} - \theta_t\right) \xrightarrow[t\to\infty]{(law)} 0,$$

since $Z_u = x_0 + Z_u^{(0)}$ and also, as we change variables $u = tv$, we use the scaling property to obtain:

$$\theta_{T^{\delta}_{\sqrt{t}}} - \theta_t \equiv \mathrm{Im}\left(\int_t^{T^{\delta}_{\sqrt{t}}} \frac{dZ_u}{Z_u}\right) \xrightarrow[t\to\infty]{(law)} \mathrm{Im}\left(\int_1^{T^{\delta}_{1}} \frac{dZ_v^{(0)}}{Z_v^{(0)}}\right).$$

Here, the limit variable is -in our opinion- of no other interest than its existence which implies (2.18), hence (2.4).

□

2.1.5 On the distributions of $T^{\theta}_c \equiv T^{\theta}_{-\infty,c}$ and $T^{\theta}_{-c,c}$

Proposition 2.1.8 *Asymptotically, for $t \to \infty$, we have:*

$$(\log t)\ \ P(T^{\theta}_c > t) \overset{t\to\infty}{\longrightarrow} (4c)/\pi \ . \tag{2.19}$$

As a consequence, for $\eta > 0$, $E[(\log T^{\theta}_c)^{\eta}_{+}] < \infty$ if and only if $\eta < 1$ (where $(\cdot)_+$ denotes the positive part).

Proof of Proposition 2.1.8 We shall rely upon the asymptotic distribution of $H_t \equiv \int_0^t \frac{ds}{|Z_s|^2}$ which is given by [ReY99]:

$$\frac{4H_t}{(\log t)^2} \underset{t\to\infty}{\overset{(law)}{\longrightarrow}} T^{\beta}_1 \equiv \inf\{t : \beta_t = 1\}, \tag{2.20}$$

or equivalently:

$$\frac{\log t}{2\sqrt{H_t}} \underset{t\to\infty}{\overset{(law)}{\longrightarrow}} |N| \,, \tag{2.21}$$

where N is a standard Gaussian variable $\mathcal{N}(0,1)$.
We note that, from representation (2.2) of θ_t, the result (2.20) is equivalent to Spitzer's theorem [Spi58]:

$$\frac{2\theta_t}{\log t} \underset{t\to\infty}{\overset{(law)}{\longrightarrow}} C_1 \overset{(law)}{=} \gamma_{T^{\beta}_1}, \tag{2.22}$$

where C_1 is a standard Cauchy variable.
We shall now use this, in order to deduce Proposition 2.1.8. We denote $S^{\theta}_t \equiv \sup_{s\le t} \theta_s \equiv S^{\gamma}_{H_t}$ and we note that (from scaling):

$$P(T^{\theta}_c \ge t) = P(S^{\gamma}_{H_t} \le c) = P(\sqrt{H_t} S^{\gamma}_1 \le c), \tag{2.23}$$

since γ and H are independent. Thus, we have (since $S^{\gamma}_1 \overset{(law)}{=} |N|$ and by making the change of variable $x = \frac{cy}{\sqrt{H_t}}$):

$$\begin{aligned} P(T^{\theta}_c \ge t) &= \sqrt{\frac{2}{\pi}} E\left[\int_0^{c/\sqrt{H_t}} dx\ e^{-\frac{x^2}{2}}\right] \\ &= \sqrt{\frac{2}{\pi}}\ c\ E\left[\int_0^1 \frac{dy}{\sqrt{H_t}} \exp\left(-\frac{c^2y^2}{2H_t}\right)\right]. \end{aligned} \tag{2.24}$$

Thus, we now deduce from (2.21) that:

$$\frac{\log t}{2} P(T_c^\theta \geq t) \overset{t\to\infty}{\longrightarrow} \sqrt{\frac{2}{\pi}}\, c\, E\left[|N|\right] = \frac{2}{\pi}\, c. \tag{2.25}$$

which is precisely (2.19).
It is now elementary to deduce from (2.25) that: for $\eta > 0$:

$$E[(\log T_c^\theta)_+^\eta] < \infty \;\Leftrightarrow\; 0 < \eta < 1,$$

since (2.25) is equivalent to:

$$u\, P(\log T_c^\theta > u) \overset{u\to\infty}{\longrightarrow} \left(\frac{4c}{\pi}\right). \tag{2.26}$$

Consequently, Fubini's theorem yields:

$$E\left[(\log T_c^\theta)_+^\eta\right] = \int_0^\infty du\, \eta\, u^{\eta-1}\; P(\log T_c^\theta > u),$$

and from (2.26) this is finite if and only if:

$$\int_.^\infty du\; u^{\eta-2} < \infty \Leftrightarrow \eta < 1.$$

So, $E[(\log T_c^\theta)_+^\eta] < \infty \;\Leftrightarrow\; 0 < \eta < 1.$

□

Now we give several examples of random times $T : C(\mathbb{R}_+, \mathbb{R}) \to \mathbb{R}_+$ which may be studied quite similarly to T_c^θ.
For such times T, it will always be true that: $H_{T(\theta)} = T(\gamma)$ is equivalent to $T(\theta) = A_{T(\gamma)}$, defined with respect to the law of Z, issued from $x_0 \neq 0$. Using Bougerol's identity, we obtain:

$$\sinh(\beta_{T(\gamma)}) \overset{(law)}{=} \hat{\beta}_{A_{T(\gamma)}} = \hat{\beta}_{(T(\theta))}. \tag{2.27}$$

where $(\hat{\beta}_u, u \geq 0)$ is a 1-dimensional Brownian motion independent of (β, γ) (or equivalently, of Z). Consequently, denoting by h_T the density of $\beta_{T(\gamma)}$, we deduce from (2.27) that:

$$E\left[\frac{1}{\sqrt{2\pi T(\theta)}} \exp(-\frac{x^2}{2T(\theta)})\right] = \frac{1}{\sqrt{1+x^2}}\; h_T(\log(x + \sqrt{1+x^2})), \tag{2.28}$$

or equivalently, changing x in $\sqrt{x}$, we obtain:

$$E\left[\frac{1}{\sqrt{2\pi T(\theta)}}\exp(-\frac{x}{2T(\theta)})\right] = \frac{1}{\sqrt{1+x}}\ h_T(\log(\sqrt{x}+\sqrt{1+x})). \quad (2.29)$$

In a number of cases, h_T is known explicitly, for example:
(i)

$$T(\gamma) = T^{\gamma}_{-d,c} \Leftrightarrow T(\theta) = \int_0^{T^{\gamma}_{-d,c}} ds\ \exp(2\beta_s) = T^{\theta}_{-d,c}.$$

So:

$$E\left[\frac{1}{\sqrt{2\pi T^{\theta}_{-d,c}}}\exp(-\frac{x}{2T^{\theta}_{-d,c}})\right] = \frac{1}{\sqrt{1+x}}\ h_{-d,c}(\log(\sqrt{x}+\sqrt{1+x})), \quad (2.30)$$

where $h_{-d,c}$ is the density of the variable $\beta_{T^{\gamma}_{-d,c}}$. The law of $\beta_{T^{\gamma}_{-d,c}}$ may be obtained from its characteristic function which is given by [ReY99], page 73:

$$\begin{aligned} E\left[\exp(i\lambda\beta_{T^{\gamma}_{-d,c}})\right] &= E\left[\exp(-\frac{\lambda^2}{2}T^{\gamma}_{-d,c})\right] \\ &= \frac{\cosh(\frac{\lambda}{2}(c-d))}{\cosh(\frac{\lambda}{2}(c+d))}. \end{aligned}$$

In particular, for $c = d$, we recover the very classical formula:

$$E\left[\exp(i\lambda\beta_{T^{\gamma}_{-c,c}})\right] = \frac{1}{\cosh(\lambda c)}.$$

It is well known that [Lev80, BiY87]:

$$\begin{aligned} E\left[\exp(i\lambda\beta_{T^{\gamma}_{-c,c}})\right] &= \frac{1}{\cosh(\lambda c)} = \frac{1}{\cosh(\pi\lambda\frac{c}{\pi})} \\ &= \int_{-\infty}^{\infty} e^{i(\frac{\lambda c}{\pi})x}\frac{1}{2\pi}\frac{1}{\cosh(\frac{x}{2})}dx \\ &\overset{y=\frac{cx}{\pi}}{=} \int_{-\infty}^{\infty} e^{i\lambda y}\frac{1}{2\pi}\frac{\frac{\pi}{c}}{\cosh(\frac{y\pi}{2c})}dy \\ &= \int_{-\infty}^{\infty} e^{i\lambda y}\frac{1}{2c}\frac{1}{\cosh(\frac{y\pi}{2c})}dy. \end{aligned} \quad (2.31)$$

So, the density of $\beta_{T^{\gamma}_{-c,c}}$ is:

$$h_{-c,c}(x) = \left(\frac{1}{2c}\right)\frac{1}{\cosh(\frac{x\pi}{2c})} = \left(\frac{1}{c}\right)\frac{1}{e^{\frac{x\pi}{2c}} + e^{-\frac{x\pi}{2c}}},$$

and

$$h_{-c,c}\left(\log(\sqrt{x}+\sqrt{1+x})\right) = \left(\frac{1}{c}\right)\frac{1}{(\sqrt{x}+\sqrt{1+x})^{\zeta} + (\sqrt{x}+\sqrt{1+x})^{-\zeta}},$$

where $\zeta = \frac{\pi}{2c}$. However using:

$$(\sqrt{x}+\sqrt{1+x})^{-\zeta} = (\sqrt{1+x}-\sqrt{x})^{\zeta}, \tag{2.32}$$

we obtain:

$$h_{-c,c}\left(\log(\sqrt{x}+\sqrt{1+x})\right) = \left(\frac{1}{c}\right)\frac{1}{(\sqrt{x}+\sqrt{1+x})^{\zeta} + (\sqrt{1+x}-\sqrt{x})^{\zeta}}. \tag{2.33}$$

So we deduce that (for $c = d$):

$$\begin{aligned} & E\left[\frac{1}{\sqrt{2\pi T^{\theta}_{-c,c}}}\exp(-\frac{x}{2T^{\theta}_{-c,c}})\right] \\ = & \left(\frac{1}{c}\right)\left(\frac{1}{\sqrt{1+x}}\right)\frac{1}{(\sqrt{x}+\sqrt{1+x})^{\zeta} + (\sqrt{1+x}-\sqrt{x})^{\zeta}}. \end{aligned} \tag{2.34}$$

□

(ii) As a second example of a random time T, let us consider the time introduced in [Val92], [ChY03], exercise 6.2, p. 178 (we use a slightly different notation). Let $(\beta_t, t \geq 0)$ be a real valued Brownian motion and define, for $c > 0$:

$$\begin{aligned} T(\theta) &\equiv T^{\hat{\theta}}_c = \inf\left\{t : \sup_{s\leq t}\theta_s - \inf_{s\leq t}\theta_s = c\right\}, \\ T(\gamma) &\equiv T^{\hat{\gamma}}_c = \inf\left\{t : \sup_{s\leq t}\gamma_s - \inf_{s\leq t}\gamma_s = c\right\}. \end{aligned}$$

Thus, from the skew-product representation (2.1), $\theta_u \equiv \gamma_{H_u}$, by replacing $u = T_c^{\hat{\theta}}$, we obtain:

$$H_{T_c^{\hat{\theta}}} = T_c^{\hat{\gamma}} \Rightarrow T_c^{\hat{\theta}} = \int_0^{T_c^{\hat{\gamma}}} ds\ \exp\left(2\beta_s\right) \equiv A_{T_c^{\hat{\gamma}}}.$$

So:

$$E\left[\frac{1}{\sqrt{2\pi T_c^{\hat{\theta}}}}\exp(-\frac{x}{2T_c^{\hat{\theta}}})\right] = \frac{1}{\sqrt{1+x}}\ h_c(\log(\sqrt{x}+\sqrt{1+x})), \tag{2.35}$$

where h_c is the density of the variable $\beta_{T_c^{\hat{\gamma}}}$. The law of $\beta_{T_c^{\hat{\gamma}}}$ may be obtained from its characteristic function which is given by [BiY87, ChY03]:

$$\begin{aligned}
E\left[\exp(i\lambda\beta_{T_c^{\hat{\gamma}}})\right] &= E\left[\exp(-\frac{\lambda^2}{2}T_c^{\hat{\gamma}})\right] = \frac{1}{(\cosh(\lambda\frac{c}{2}))^2} = \frac{1}{\left(\cosh(\pi\lambda\frac{c}{2\pi})\right)^2} \\
&= \int_{-\infty}^{\infty} e^{i(\frac{\lambda c}{2\pi})x}\frac{1}{2\pi}\frac{x}{\sinh(\frac{x}{2})}dx \\
&\overset{y=\frac{cx}{2\pi}}{=} \int_{-\infty}^{\infty} e^{i\lambda y}\frac{1}{2\pi}\frac{\frac{2\pi y}{c}}{\sinh(\frac{\pi y}{c})}\frac{2\pi}{c}dy \\
&= \int_{-\infty}^{\infty} e^{i\lambda y}\frac{2\pi}{c^2}\frac{y}{\sinh(\frac{\pi y}{c})}dy.
\end{aligned} \tag{2.36}$$

So, the density of $\beta_{T_c^{\hat{\gamma}}}$ is:

$$h_c(y) = \left(\frac{2\pi y}{c^2}\right)\frac{1}{\sinh(\frac{\pi y}{c})} = \frac{4\pi}{c^2}\frac{y}{e^{\frac{\pi y}{c}} - e^{-\frac{\pi y}{c}}},$$

and

$$h_c\left(\log(\sqrt{x}+\sqrt{1+x})\right) = \frac{4\pi}{c^2}\frac{\log(\sqrt{x}+\sqrt{1+x})}{(\sqrt{x}+\sqrt{1+x})^{\hat{\zeta}} - (\sqrt{x}+\sqrt{1+x})^{-\hat{\zeta}}},$$

where $\hat{\zeta} = \frac{\pi}{c}$. Thus:

$$\begin{aligned}
&E\left[\frac{1}{\sqrt{2\pi T_c^{\hat{\theta}}}}\exp(-\frac{x}{2T_c^{\hat{\theta}}})\right] \\
&= \frac{4\pi}{c^2}\frac{1}{\sqrt{1+x}}\frac{\log(\sqrt{x}+\sqrt{1+x})}{(\sqrt{x}+\sqrt{1+x})^{\hat{\zeta}} - (\sqrt{1+x}-\sqrt{x})^{\hat{\zeta}}}.
\end{aligned} \tag{2.37}$$

We note that this study may be related to [PiY03]; and more precisely $\beta_{T_c^{\hat{\gamma}}}$ and $T_c^{\hat{\gamma}}$ correspond to the variables C_2 and $\hat{C}_2$ respectively (see e.g. Table 6 in p. 312).

□

Let us now return to the case of $T^{\theta}_{-c,c}$ (example (i)). More specifically, we shall obtain its density function $f(t)$.

Proposition 2.1.9 *The density function f of $T^{\theta}_{-c,c}$ is given by:*

$$f(t) = \frac{1}{\sqrt{2}c} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\nu_k)}{\Gamma(2\nu_k)} \frac{1}{\sqrt{t}} e^{-\frac{1}{4t}} M_{\frac{1}{2},\nu_k}(\frac{1}{2t}), \tag{2.38}$$

where $M_{a,b}(\cdot)$ is the Whittaker function with parameters a, b. Equivalently:

$$f(t) = \frac{\sqrt{2}}{c} \sum_{k=0}^{\infty} (-1)^k \frac{1}{\sqrt{t}} e^{-\frac{1}{2t}} \left(\frac{1}{2t}\right)^{\nu_k+\frac{1}{2}} \nu_k \sum_{n=0}^{\infty} \frac{\Gamma(\nu_k+n)}{\Gamma(2\nu_k+n+1)} \frac{1}{n!} \left(\frac{1}{2t}\right)^n, \tag{2.39}$$

where $\nu_k = \frac{\pi}{4c}(2k+1)$.

Proof of Proposition 2.1.9 The following calculation relies upon a private note by A. Comtet [Com06]. We denote:

$$\varphi_\zeta(x) = (\sqrt{x} + \sqrt{1+x})^\zeta + (\sqrt{1+x} - \sqrt{x})^\zeta.$$

Noting:

$$\sqrt{1+x} = \cosh \frac{y}{2} \Longleftrightarrow y = 2 \arg\cosh(\sqrt{1+x}), \tag{2.40}$$

we get:

$$\begin{aligned} \varphi_\zeta(x) &= (\sinh \frac{y}{2} + \cosh \frac{y}{2})^\zeta + (\cosh \frac{y}{2} - \sinh \frac{y}{2})^\zeta \\ &= 2 \cosh \frac{y\zeta}{2}. \end{aligned}$$

So, from (2.34), we have:

$$II := E\left[\frac{1}{\sqrt{2\pi T^{\theta}_{-c,c}}} \exp(-\frac{x}{2T^{\theta}_{-c,c}})\right] = \frac{1}{\psi} \frac{1}{\cosh \frac{y}{2}} \frac{1}{\cosh \frac{\pi y}{2\psi}}, \tag{2.41}$$

where $\psi = 2c$. However, expanding $\cosh \frac{\pi y}{2\psi}$, we get:

$$\frac{1}{\cosh \frac{\pi y}{2\psi}} = 2\frac{e^{-\frac{\pi y}{2\psi}}}{1+e^{-\frac{\pi y}{\psi}}} = 2\sum_{k=0}^{\infty}\left(-e^{-\frac{\pi y}{\psi}}\right)^k e^{-\frac{\pi y}{2\psi}},$$

and from (2.41), we deduce that:

$$\begin{aligned} II &= \sum_{k=0}^{\infty} \frac{2}{\psi}\frac{(-1)^k}{\cosh\frac{y}{2}} e^{-\frac{\pi}{2\psi}(2k+1)y} \\ &= \sum_{k=0}^{\infty} \frac{4(-1)^k}{\psi\sqrt{2}\sqrt{2\sinh\frac{y}{2}\cosh\frac{y}{2}}}\sqrt{\frac{\sinh\frac{y}{2}}{\cosh\frac{y}{2}}} e^{-\nu_k y} \\ &= \sum_{k=0}^{\infty} \frac{4(-1)^k}{\psi\sqrt{2}\sqrt{2\sinh\frac{y}{2}\cosh\frac{y}{2}}}\sqrt{\tanh\frac{y}{2}} e^{-\nu_k y}, \end{aligned}$$

where $\nu_k = \frac{\pi}{2\psi}(2k+1)$.
From (2.40), we have $1+x = \cosh^2\frac{y}{2} \Longleftrightarrow x = \sinh^2\frac{y}{2}$, thus:

$$(\tanh\frac{y}{2})^{1/2} = \sqrt{\frac{\sinh\frac{y}{2}}{\cosh\frac{y}{2}}} = \left(\frac{\sqrt{x}}{\sqrt{1+x}}\right)^{1/2} = \left(\frac{x}{1+x}\right)^{1/4}.$$

Moreover, we know that ([AbSt70], equation 8.6.10, or [Leb63]):

$$i\sqrt{\frac{\pi}{2\sinh y}} e^{-\nu_k y} = Q^{1/2}_{\nu_k - 1/2}(\cosh y),$$

where $\{Q^a_b(\cdot)\}$ is the family of Legendre functions and $\cosh y = 2x+1$. So, we deduce:

$$II = \sum_{k=0}^{\infty} \frac{4(-i)}{\psi\sqrt{\pi}}(-1)^k \left(\frac{x}{1+x}\right)^{1/4} Q^{1/2}_{\nu_k-1/2}(2x+1). \tag{2.42}$$

By using formula 7.621.9, page 864 [GrR65]:

$$\int_0^{\infty} e^{-sw} M_{l,\nu_k}(w)\frac{dw}{w} = \frac{2\Gamma(1+2\nu_k)\, e^{-i\pi l}}{\Gamma(\frac{1}{2}+\nu_k+l)}\left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^{l/2} Q^l_{\nu_k-1/2}(2s), \tag{2.43}$$

with: $l=\frac{1}{2}$, $\nu_k=\frac{\pi}{2\psi}(2k+1)$, $s=x+\frac{1}{2}$ and $M_{\cdot,\cdot}(\cdot)$ denoting the Whittaker function, which is defined as:

$$M_{a,b}(w)=w^{b+\frac{1}{2}}e^{-\frac{1}{2}w}\frac{\Gamma(2b+1)}{\Gamma(\frac{1}{2}+b-a)}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{1}{2}+b-a+n)}{\Gamma(2b+1+n)}\frac{w^n}{n!}.$$

we have:

$$-2i\frac{\Gamma(1+2\nu_k)}{\Gamma(1+\nu_k)}\left(\frac{x}{1+x}\right)^{1/4}Q^{1/2}_{\nu_k-1/2}(2x+1)=\int_0^\infty e^{-sw}M_{1/2,\nu_k}(w)\frac{dw}{w}. \tag{2.44}$$

From (2.42) and by changing the variable $w=\frac{1}{2t}$, we deduce:

$$\begin{aligned} II &= \sum_{k=0}^{\infty}\frac{2}{\psi\sqrt{\pi}}(-1)^k\frac{\Gamma(\nu_k+1)}{\Gamma(2\nu_k+1)}\int_0^\infty\frac{dw}{w}\exp\left(-w(x+\frac{1}{2})\right)M_{1/2,\nu_k}(w) \\ &= \sum_{k=0}^{\infty}\int_0^\infty\frac{dt}{t}\frac{2}{\psi\sqrt{\pi}}(-1)^k\frac{\Gamma(\nu_k+1)}{\Gamma(2\nu_k+1)}\exp\left(-\frac{1}{4t}-\frac{x}{2t}\right)M_{1/2,\nu_k}(\frac{1}{2t}). \end{aligned} \tag{2.45}$$

By using the equations (2.41) and (2.45), we conclude:

$$\begin{aligned} & E\left[\frac{1}{\sqrt{2\pi T^{\theta}_{-c,c}}}\exp\left(-\frac{x}{2T^{\theta}_{-c,c}}\right)\right] \\ = &\sum_{k=0}^{\infty}\int_0^\infty\frac{dt}{t}\frac{2}{\psi\sqrt{\pi}}(-1)^k\frac{\Gamma(\nu_k+1)}{\Gamma(2\nu_k+1)}\exp\left(-\frac{1}{4t}-\frac{x}{2t}\right)M_{\frac{1}{2},\nu_k}(\frac{1}{2t}) \\ = &\sum_{k=0}^{\infty}\int_0^\infty\frac{dt}{t}\frac{2}{\psi\sqrt{\pi}}(-1)^k\frac{\Gamma(\frac{\pi}{4c}(2k+1)+1)}{\Gamma(2\frac{\pi}{4c}(2k+1)+1)}\exp\left(-\frac{1}{4t}-\frac{x}{2t}\right)M_{\frac{1}{2},\frac{\pi}{4c}(2k+1)}(\frac{1}{2t}). \end{aligned} \tag{2.46}$$

Thus, the density function f of $T^{\theta}_{-c,c}$ is given by:

$$\begin{aligned} f(t) &= \frac{2\sqrt{2}}{\psi}\sum_{k=0}^{\infty}(-1)^k \frac{\Gamma(\nu_k+1)}{\Gamma(2\nu_k+1)}\frac{1}{\sqrt{t}}e^{-\frac{1}{4t}}M_{\frac{1}{2},\nu_k}(\frac{1}{2t}) && (2.47)\\ &= \frac{\sqrt{2}}{c}\sum_{k=0}^{\infty}(-1)^k \frac{\Gamma(\frac{\pi}{4a}(2k+1)+1)}{\Gamma(\frac{\pi}{2a}(2k+1)+1)}\frac{1}{\sqrt{t}}e^{-\frac{1}{4t}}M_{\frac{1}{2},\frac{\pi}{4a}(2k+1)}(\frac{1}{2t}) && (2.48)\\ &= \frac{\sqrt{2}}{c}\sum_{k=0}^{\infty}(-1)^k \frac{\nu_k\Gamma(\nu_k)}{2\nu_k\Gamma(2\nu_k)}\frac{1}{\sqrt{t}}e^{-\frac{1}{4t}}M_{\frac{1}{2},\nu_k}(\frac{1}{2t}), && (2.49)\end{aligned}$$

where the Whittaker function $M_{\frac{1}{2},\nu_k}(\frac{1}{2t})$ is:

$$\begin{aligned} & M_{\frac{1}{2},\frac{\pi}{4c}(2k+1)}(\frac{1}{2t})\\ = & \left(\frac{1}{2t}\right)^{\frac{\pi}{4c}(2k+1)+\frac{1}{2}} e^{-\frac{1}{4t}}\frac{\Gamma(\frac{\pi}{2c}(2k+1)+1)}{\Gamma(\frac{\pi}{4c}(2k+1))}\sum_{n=0}^{\infty}\frac{\Gamma(\frac{\pi}{4c}(2k+1)+n)}{\Gamma(\frac{\pi}{2c}(2k+1)+1+n)}\frac{1}{n!}\left(\frac{1}{2t}\right)^n\\ = & \left(\frac{1}{2t}\right)^{\nu_k+\frac{1}{2}} e^{-\frac{1}{4t}}\frac{\Gamma(2\nu_k+1)}{\Gamma(\nu_k)}\sum_{n=0}^{\infty}\frac{\Gamma(\nu_k+n)}{\Gamma(2\nu_k+1+n)}\frac{1}{n!}\left(\frac{1}{2t}\right)^n\\ = & \left(\frac{1}{2t}\right)^{\nu_k+\frac{1}{2}} e^{-\frac{1}{4t}}(2\nu_k)\frac{\Gamma(2\nu_k)}{\Gamma(\nu_k)}\sum_{n=0}^{\infty}\frac{\Gamma(\nu_k+n)}{(2\nu_k+n)\Gamma(2\nu_k+n)}\frac{1}{n!}\left(\frac{1}{2t}\right)^n. \qquad (2.50)\end{aligned}$$

Thus, from (2.49) and (2.50), we deduce (2.39).

□

Next, we present the graphs of different approximations $f_{K,N}(t)$ of $f(t)$, in (2.39), where $f_{K,N}$ denotes the sum in the series in (2.39) of the terms for $k \leq K$, and $n \leq N$.

Remark 2.1.10

- *Figure 2.2 represents the approximation of the density function f with respect to the time t (for K and $N \leq 9$), with $c = 2\pi$, whereas Figure 2.3 represents the approximation of f with respect to the time t for several values of K and N, with $c = 2\pi$.*
- *From Figure 2.3, we may remark that the approximation K and $N \leq 9$ is sufficiently good (comparing to the one for K and $N \leq 100$).*

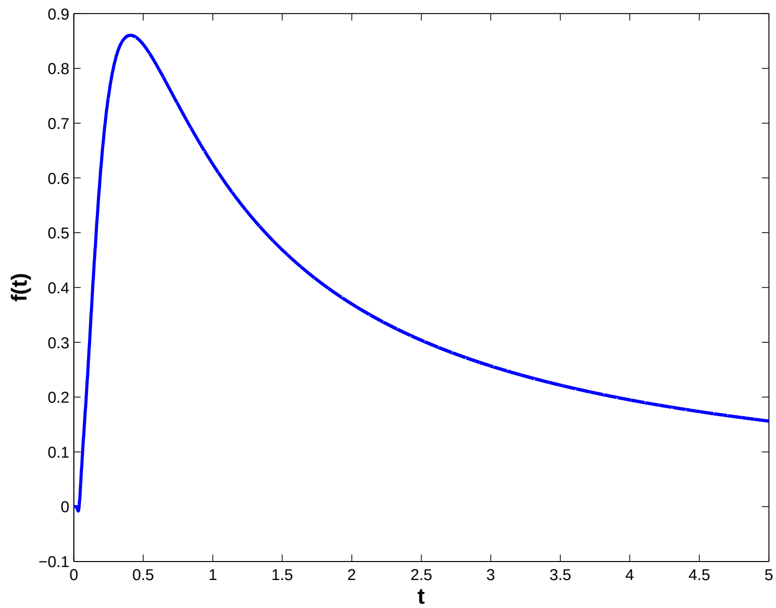


Figure 2.2: **Graph of $f_{9,9}(t)$, with $c = 2\pi$.**

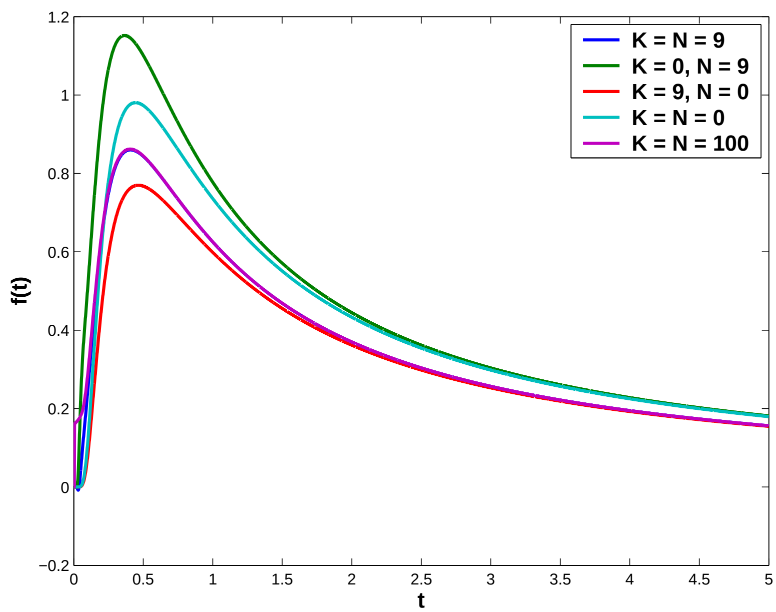


Figure 2.3: **Graph of $f_{K,N}(t)$ for several values of K and N, with $c = 2\pi$.**

- *For the case* K *and* $N \leq 9$ *it seems that locally, in a small area around 0,* $f(t) < 0$ *which is not right. This is due to the first negative (*$k = 1$*) term of the sum and due to the fact that we have omitted many terms. However, this is not a problem because it appears only locally. Similar irregularities have already been observed in previous articles [Ish05] p.275.*

2.1.6 On the first moment of $\ln\left(T^{\theta}_{-c,c}\right)$

This subsection is related to a result in [CMY98].

Proposition 2.1.11 *The first moment of* $\ln\left(T^{\theta}_{-c,c}\right)$ *has the following integral representation:*

$$E\left[\ln\left(T^{\theta}_{-c,c}\right)\right] = 2\int_0^{\infty} \frac{dz}{\cosh\left(\frac{\pi z}{2}\right)} \ln\left(\sinh\left(cz\right)\right) + \ln\left(2\right) + c_E, \tag{2.51}$$

where $c_E = -\Gamma'(1)$ *is the Euler-Mascheroni constant (also called Euler's constant).*

Proof of Proposition 2.1.11 Let us return to equations (2.2) and (2.6). So, for $t = T^{\theta}_{-c,c}$, we have:

$$\theta_{T^{\theta}_{-c,c}} = \gamma_{H_{T^{\theta}_{-c,c}}} \Longleftrightarrow H_{T^{\theta}_{-c,c}} = T^{\gamma}_{-c,c} \Longleftrightarrow T^{\theta}_{-c,c} = A_{T^{\gamma}_{-c,c}}. \tag{2.52}$$

Thus, for $\varepsilon > 0$:

$$E\left[\left(T^{\theta}_{-c,c}\right)^{\varepsilon}\right] = E\left[\left(A_{T^{\gamma}_{-c,c}}\right)^{\varepsilon}\right].$$

Consider $(\delta_t, t \geq 0)$ a Brownian motion, independent of A_t. Then, Bougerol's identity and the scaling property yield ($\mathcal{G}_a$ denotes a gamma variable with parameter a, and $N^2 \overset{(law)}{=} 2\mathcal{G}_{1/2}$):

$$\begin{aligned} E\left[|\sinh\left(B_t\right)|^{2\varepsilon}\right] &= E\left[|\delta_{A_t}|^{2\varepsilon}\right] = E\left[\left(A_t^{\varepsilon}\right)|\delta_1|^{2\varepsilon}\right] \\ &= E\left[A_t^{\varepsilon}\right]\, E\left[\left(2\mathcal{G}_{1/2}\right)^{\varepsilon}\right] \\ &= E\left[A_t^{\varepsilon}\right]\, \left(2^{\varepsilon}\right)\, \frac{\Gamma\left(\frac{1}{2}+\varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)}, \end{aligned}$$

because:

$$E\left[\left(\mathcal{G}_{1/2}\right)^{\varepsilon}\right] = \int_0^{\infty} x^{\varepsilon+\frac{1}{2}-1} \frac{e^{-x}}{\Gamma\left(\frac{1}{2}\right)}\, dx = \frac{\Gamma\left(\frac{1}{2}+\varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)}.$$

Thus, for $t = T^{\gamma}_{-c,c}$, we have:

$$E\left[\left|\sinh\left(B_{T^{\gamma}_{-c,c}}\right)\right|^{2\varepsilon}\right] = E\left[A^{\varepsilon}_{T^{\gamma}_{-c,c}}\right]\, \left(2^{\varepsilon}\right)\, \frac{\Gamma\left(\frac{1}{2}+\varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)}. \tag{2.53}$$

Recall that [Lev80, BiY87]:

$$E\left[\exp(i\lambda B_{T^{\gamma}_{-c,c}}))\right] = E\left[\exp(-\frac{\lambda^2}{2}T^{\gamma}_{-c,c}))\right] = \frac{1}{\cosh(\lambda c)},$$

and the density of $B_{T^{\gamma}_{-c,c}}$ is:

$$h_{-c,c}(y) = \left(\frac{1}{2c}\right)\frac{1}{\cosh(\frac{y\pi}{2c})} = \left(\frac{1}{c}\right)\frac{1}{e^{\frac{y\pi}{2c}} + e^{-\frac{y\pi}{2c}}}.$$

Thus, on the left hand side of (2.53), we have:

$$\begin{aligned} E\left[\left|\sinh\left(B_{T^{\gamma}_{-c,c}}\right)\right|^{2\varepsilon}\right] &= \int_{-\infty}^{\infty}\frac{dy}{2c}\,\frac{1}{\cosh(\frac{\pi y}{2c})}\left|\sinh(y)\right|^{2\varepsilon} \\ &= \int_0^{\infty}\frac{dy}{c}\,\frac{1}{\cosh(\frac{\pi y}{2c})}\left(\sinh y\right)^{2\varepsilon} \\ &= \int_0^{\infty}dz\,\frac{1}{\cosh(\frac{\pi z}{2})}\left(\sinh(cz)\right)^{2\varepsilon}, \end{aligned}$$

where we have made the change of variable $z = \frac{y}{c}$. So, from (2.53), by writing:

$$E\left[A^{\varepsilon}_{T^{\gamma}_{-c,c}}\right] = E\left[\left(T^{\theta}_{-c,c}\right)^{\varepsilon}\right] = E\left[e^{\varepsilon\,\ln\left(T^{\theta}_{-c,c}\right)}\right],$$

we deduce:

$$\frac{\Gamma\left(\frac{1}{2}+\varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)}E\left[e^{\varepsilon\,\ln\left(T^{\theta}_{-c,c}\right)}\right] = \frac{1}{2^{\varepsilon}}\int_0^{\infty}\frac{dz}{\cosh(\frac{\pi z}{2})}\left(\sinh(cz)\right)^{2\varepsilon},$$

and by removing 1 from both sides, we obtain:

$$\frac{\Gamma\left(\frac{1}{2}+\varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)}E\left[e^{\varepsilon\,\ln\left(T^{\theta}_{-c,c}\right)}\right] - 1 = \int_0^{\infty}\frac{dz}{\cosh(\frac{\pi z}{2})}\left(\frac{\left(\sinh(cz)\right)^{2\varepsilon}}{2^{\varepsilon}} - 1\right). \quad (2.54)$$

On the left hand side, we apply the trivial identity $ab - 1 = a(b-1) + a - 1$ with $a = \frac{\Gamma\left(\frac{1}{2}+\varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)}$ and $b = E\left[e^{\varepsilon\,\ln\left(T^{\theta}_{-c,c}\right)}\right]$, we divide by ε and we take the limit for $\varepsilon \to 0$. Thus:

$$\begin{aligned} \frac{a(b-1)}{\varepsilon} &= \frac{\Gamma\left(\frac{1}{2}+\varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)}\frac{E\left[e^{\varepsilon\,\ln\left(T^{\theta}_{-c,c}\right)}\right] - 1}{\varepsilon} \\ &\overset{\varepsilon\to 0}{\longrightarrow} E\left[\ln\left(T^{\theta}_{-c,c}\right)\right], \end{aligned}$$

and:

$$\begin{aligned}\frac{a-1}{\varepsilon} &= \frac{1}{\varepsilon}\left(\frac{\Gamma\left(\frac{1}{2}+\varepsilon\right)}{\Gamma\left(\frac{1}{2}\right)}-1\right)=\frac{1}{\Gamma\left(\frac{1}{2}\right)}\left(\frac{\Gamma\left(\frac{1}{2}+\varepsilon\right)-\Gamma\left(\frac{1}{2}\right)}{\varepsilon}\right)\\ &\xrightarrow{\varepsilon\to 0} \frac{1}{\sqrt{\pi}}\Gamma'\left(\frac{1}{2}\right)=\frac{1}{\sqrt{\pi}}\left(-\sqrt{\pi}\right)\left(c_E+2\ln 2\right)=-\left(c_E+2\ln 2\right).\end{aligned}$$

On the right hand side of (2.54), we have:

$$\frac{1}{\varepsilon}\left[\left(\frac{(\sinh(cz))^2}{2}\right)^{\varepsilon}-1\right]=\frac{1}{\varepsilon}\left[\exp\left(\varepsilon\,\ln\left(\frac{(\sinh(cz))^2}{2}\right)\right)-1\right],$$

hence:

$$\begin{aligned}&\frac{1}{\varepsilon}\int_0^\infty \frac{dz}{\cosh(\frac{\pi z}{2})}\left(\frac{(\sinh(cz))^{2\varepsilon}}{2^\varepsilon}-1\right)\\ \xrightarrow{\varepsilon\to 0}\ &\int_0^\infty \frac{dz}{\cosh(\frac{\pi z}{2})}\ln\left(\frac{(\sinh(cz))^2}{2}\right)\\ =\ & 2\int_0^\infty \frac{dz}{\cosh(\frac{\pi z}{2})}\left(\ln\left(\sinh(cz)\right)\right)-\ln\left(2\right),\end{aligned}$$

which finishes the proof.

$\square$

2.2 The Ornstein-Uhlenbeck case

2.2.1 An identity in law for Ornstein-Uhlenbeck processes, which is connected to Bougerol's identity

Consider the complex valued Ornstein-Uhlenbeck (OU) process:

$$Z_t = z_0 + \tilde{Z}_t - \lambda \int_0^t Z_s ds, \tag{2.55}$$

where $\tilde{Z}_t$ is a complex valued Brownian motion (BM), $z_0 \in \mathbb{C}$ and $\lambda \geq 0$ and $T_c^{(\lambda)} \equiv T_c^{\theta^Z} \equiv \inf\left\{t \geq 0 : \left|\theta_t^Z\right| = c\right\}$ (θ_t^Z is the continuous winding process associated to Z) denoting the first hitting time of the symmetric conic boundary of angle c for Z. It is well known that [ReY99]:

$$\begin{aligned} Z_t &= e^{-\lambda t}\left(z_0 + \int_0^t e^{\lambda s} d\tilde{Z}_s\right) \\ &= e^{-\lambda t}\left(\mathbb{B}_{\alpha_t}\right), \end{aligned} \tag{2.56}$$

where, in the second equation, with the help of Dambis-Dubins-Schwarz Theorem, $(\mathbb{B}_t, t \geq 0)$ is a complex valued Brownian motion starting from z_0 and:

$$\alpha_t = \int_0^t e^{2\lambda s} ds = \frac{e^{2\lambda t} - 1}{2\lambda} \quad .$$

We are interested in the study of the continuous winding process $\theta_t^Z = \mathrm{Im}(\int_0^t \frac{dZ_s}{Z_s}), t \geq 0$. By applying Itô's formula to (2.56), we have:

$$dZ_s = e^{-\lambda s}(-\lambda)\mathbb{B}_{\alpha_s} ds + e^{-\lambda s} d\left(\mathbb{B}_{\alpha_s}\right).$$

We divide by Z_s and we obtain:

$$\frac{dZ_s}{Z_s} = (-\lambda)ds + \frac{d\mathbb{B}_{\alpha_s}}{\mathbb{B}_{\alpha_s}},$$

and so:

$$\mathrm{Im}\left(\frac{dZ_s}{Z_s}\right) = \mathrm{Im}\left(\frac{d\mathbb{B}_{\alpha_s}}{\mathbb{B}_{\alpha_s}}\right),$$

which means that:

$$\theta_t^Z = \theta_{\alpha_t}^{\mathbb{B}} \quad .$$

Thus, the following holds:

Proposition 2.2.1 *Using the notation which we introduced before, we have:*

$$\theta^Z_t = \theta^{\mathbb{B}}_{\alpha_t}, \tag{2.57}$$

and:

$$T^{(\lambda)}_c = \frac{1}{2\lambda} \ln \left(1 + 2\lambda T^{\theta^{\mathbb{B}}}_{-c,c}\right), \tag{2.58}$$

where $T^{\theta^{\mathbb{B}}}_{-c,c}$ is the exit time from a cone of angle c for the complex valued BM $\mathbb{B}$.

Proof of Proposition 2.2.1 We define

$$\begin{aligned} T^{(\lambda)}_c &\equiv T^{\theta^Z}_c \equiv \inf\left\{t \geq 0 : \left|\theta^Z_t\right| = c\right\} \\ &= \inf\left\{t \geq 0 : \left|\theta^{\mathbb{B}}_{\alpha_t}\right| = c\right\}. \end{aligned} \tag{2.59}$$

Thus, we deduce that $\alpha_{T^{(\lambda)}_c} = T^{\theta^{\mathbb{B}}}_c \equiv T^{\theta}_{-c,c}$. However, $T^{\theta}_{-c,c}$ (the exit time from a cone for the BM) has already been studied in the previous chapter and we know the explicit formula of its density function (Proposition 2.1.9). So:

$$T^{(\lambda)}_c = \alpha^{-1}\left(T^{\theta^{\mathbb{B}}}_c\right) = \alpha^{-1}\left(T^{\theta}_{-c,c}\right), \tag{2.60}$$

where $\alpha^{-1}(t) = \frac{1}{2\lambda} \ln(1 + 2\lambda t)$. Consequently:

$$T^{(\lambda)}_c = \frac{1}{2\lambda} \ln\left(1 + 2\lambda T^{\theta}_{-c,c}\right),$$

and:

$$E\left[T^{(\lambda)}_c\right] = \frac{1}{2\lambda} E\left[\ln\left(1 + 2\lambda T^{\theta}_{-c,c}\right)\right], \tag{2.61}$$

which finishes the proof.

□

From now on, for simplicity, we shall take $z_0 = 1$ (but this is really no restriction, as the dependency in z_0, which is exhibited in (2.7), is very simple). The following Proposition may be considered as an extension of the identity in law (ii) in Proposition 2.1.4, which results from Bougerol's identity.

Proposition 2.2.2 *Consider $(Z_t^\lambda, t \geq 0)$ and $(U_t^\lambda, t \geq 0)$ two independent Ornstein-Uhlenbeck processes, the first one complex valued and the second one real valued, both starting from a point different from 0, and call $T_b^{(\lambda)}(U^\lambda) = \inf\left\{t \geq 0 : e^{\lambda t} U_t^\lambda = b\right\}$, for any $b \geq 0$. Then, an Ornstein-Uhlenbeck extension of identity in law (ii) in Proposition 2.1.4 is the following:*

$$\theta^{Z^\lambda}_{T_b^{(\lambda)}(U^\lambda)} \overset{(law)}{=} C_{a(b)}, \tag{2.62}$$

where $a(x) = \arg\sinh(x)$.

Proof of Proposition 2.2.2 Let us consider a second Ornstein-Uhlenbeck process $(U_t^\lambda, t \geq 0)$ independent of the first one. Then, taking equation (2.56) for U_t^λ, we have:

$$e^{\lambda t} U_t^\lambda = \delta_{\left(\frac{e^{2\lambda t}-1}{2\lambda}\right)}, \tag{2.63}$$

where $(\delta_t, t \geq 0)$ is a complex valued Brownian motion starting from $z_0 = 1$. So:

$$T_b^{(\lambda)}(U^\lambda) = \frac{1}{2\lambda} \ln\left(1 + 2\lambda T_b^\delta\right). \tag{2.64}$$

Equation (2.57) for $t = \frac{1}{2\lambda} \ln\left(1 + 2\lambda T_b^\delta\right)$, equivalently: $\alpha(t) = T_b^\delta$ becomes (we suppose that $z_0 = 1$):

$$\theta^{Z^\lambda}_{T_b^{(\lambda)}(U^\lambda)} = \theta^{Z^\lambda}_{\frac{1}{2\lambda}\ln\left(1+2\lambda T_b^\delta\right)} = \theta^{\mathbb{B}}_{u=T_b^\delta} \overset{(law)}{=} C_{a(b)}.$$

□

2.2.2 On the distribution of $T^\theta_{-c,c}$ for an Ornstein-Uhlenbeck process

Now we turn to the study of the density function of

$$T_c^{(\lambda)} \equiv T_c^{\theta^Z} \equiv \inf\left\{t \geq 0 : \left|\theta_t^Z\right| = c\right\},$$

and its first moment.

Proposition 2.2.3 *Asymptotically for λ large, for $z_0 = 1$, we have:*

$$2\lambda\, E\left[T_c^{(\lambda)}\right] - \ln(2\lambda) \overset{\lambda\to\infty}{\longrightarrow} E\left[\ln\left(T^{\theta}_{-c,c}\right)\right], \tag{2.65}$$

and:

$$E\left[\ln\left(T^{\theta}_{-c,c}\right)\right] = 2\int_0^{\infty} \frac{dz}{\cosh\left(\frac{\pi z}{2}\right)} \ln\left(\sinh(cz)\right) + \ln(2) + c_E, \tag{2.66}$$

where c_E is Euler's constant.
For $c < \frac{\pi}{8}$, we have the asymptotic equivalence:

$$\frac{1}{\lambda}\left(E\left[T_c^{(\lambda)}\right] - E\left[|\sinh\left(B_{T^{\gamma}_{-c,c}}\right)|^2\right]\right) \overset{\lambda\to 0}{\longrightarrow} -\frac{1}{3}E\left[|\sinh\left(B_{T^{\gamma}_{-c,c}}\right)|^4\right]. \tag{2.67}$$

Equivalently:

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} E\left[T_c^{(\lambda)}\right] = \lim_{\lambda\to 0}\left[\frac{1}{\lambda}\left(E\left[T_c^{(\lambda)}\right] - E\left[T_c^{(0)}\right]\right)\right] = -\frac{1}{3}E\left[|\sinh\left(B_{T^{\gamma}_{-c,c}}\right)|^4\right]. \tag{2.68}$$

Moreover:

$$E\left[|\sinh\left(B_{T^{\gamma}_{-c,c}}\right)|^4\right] = \int_0^{\infty} \frac{dz}{\cosh\left(\frac{\pi z}{2}\right)} \left(\sinh(cz)\right)^4. \tag{2.69}$$

More precisely, for $c < \frac{\pi}{8}$:

$$E\left[|\sinh\left(B_{T^{\gamma}_{-c,c}}\right)|^4\right] = \frac{1}{8}\left(\frac{1}{\cos(4c)} - 4\frac{1}{\cos(2c)} + 3\right), \tag{2.70}$$

and asymptotically:

$$E\left[|\sinh\left(B_{T^{\gamma}_{-c,c}}\right)|^4\right] \underset{c\to 0}{\simeq} 5c^4. \tag{2.71}$$

Proof of Proposition 2.2.3
<u>λ large</u>
Let us return to equation (2.61). For $\lambda \to \infty$, we have:

$$\begin{aligned} E\left[T_c^{(\lambda)}\right] &= \frac{1}{2\lambda}E\left[\ln\left(1 + 2\lambda T^{\theta}_{-c,c}\right)\right] \\ &= \frac{1}{2\lambda}E\left[\ln\left(2\lambda\left(T^{\theta}_{-c,c} + \frac{1}{2\lambda}\right)\right)\right] \\ &= \frac{\ln(2\lambda)}{2\lambda} + \frac{1}{2\lambda}E\left[\ln\left(T^{\theta}_{-c,c} + \frac{1}{2\lambda}\right)\right]. \end{aligned}$$

Thus:

$$2\lambda\, E\left[T_c^{(\lambda)}\right] - \ln(2\lambda) \stackrel{\lambda\to\infty}{\longrightarrow} E\left[\ln\left(T^{\theta}_{-c,c}\right)\right],$$

which is precisely (2.65). Moreover, by the integral representation (2.51) for $E\left[\ln\left(T^{\theta}_{-c,c}\right)\right]$, we deduce (2.66).

λ small

We shall now study the case $\lambda \to 0$. We have that:

$$T_c^{(\lambda)} = \frac{1}{2\lambda}\ln\left(1 + 2\lambda T^{\theta}_{-c,c}\right).$$

For $c < \frac{\pi}{8}$, from Spitzer (2.3), (at least) the first two positive moments of $T^{\theta}_{-c,c}$ are finite: $E\left[\left(T^{\theta}_{-c,c}\right)^p\right] < \infty$, $(p = 1, 2)$. We make the elementary computation:

$$\begin{aligned}\frac{1}{\lambda}\left(\frac{\ln(1+2\lambda x)}{2\lambda} - x\right) &= \frac{1}{\lambda}\left(\frac{1}{2\lambda}\int_1^{1+2\lambda x}\frac{dy}{y} - x\right)\\ &\stackrel{y=1+a}{=} \frac{1}{2\lambda^2}\int_0^{2\lambda x}\left(\frac{1}{1+a} - 1\right)\,da \stackrel{a=2\lambda b}{=} -2\int_0^x \frac{b\,db}{1+2\lambda b} \stackrel{\lambda\to 0}{\longrightarrow} -x^2.\end{aligned}$$

Consequently, by replacing $x = T^{\theta}_{-c,c}$, we have:

$$\frac{1}{\lambda}\left(E\left[T_c^{(\lambda)}\right] - E\left[T^{\theta}_{-c,c}\right]\right) = E\left[-2\int_0^{T^{\theta}_{-c,c}}\frac{b\,db}{1+2\lambda b}\right].$$

We may now use the dominated convergence theorem [Bil78], since the (db) integral is majorized by $(T^{\theta}_{-c,c})^2$, which is integrable. Thus:

$$\frac{1}{\lambda}\left(E\left[T_c^{(\lambda)}\right] - E\left[T^{\theta}_{-c,c}\right]\right) \stackrel{\lambda\to 0}{\longrightarrow} -E\left[(T^{\theta}_{-c,c})^2\right].$$

Following the proof of Proposition 2.1.11, Bougerol's identity and the scaling property yield:

$$\begin{aligned}E\left[(\sinh(B_t))^2\right] &= E\left[(\delta_{A_t})^2\right] = E\left[(A_t)(\delta_1)^2\right] = E[A_t]\ E\left[(\delta_1)^2\right]\\ &= E[A_t].\end{aligned}$$

Thus, for $t = T^{\gamma}_{-c,c}$, we have:

$$E\left[A_{T^{\gamma}_{-c,c}}\right] = E\left[\left|\sinh\left(B_{T^{\gamma}_{-c,c}}\right)\right|^2\right].$$

Similarly:

$$\begin{aligned} E\left[(\sinh(B_t))^4\right] &= E\left[(\delta_{A_t})^4\right] = E\left[(A_t)^2(\delta_1)^4\right] = E\left[(A_t)^2\right]\, E\left[(\delta_1)^4\right] \\ &= 3E\left[(A_t)^2\right]. \end{aligned}$$

Thus, for $t = T^{\gamma}_{-c,c}$, we have:

$$E\left[\left(A_{T^{\gamma}_{-c,c}}\right)^2\right] = \frac{1}{3}E\left[|\sinh\left(B_{T^{\gamma}_{-c,c}}\right)|^4\right].$$

So, because $A_{T^{\gamma}_{-c,c}} = T^{\theta}_{-c,c}$, we deduce (2.67). In order to prove (2.68), it suffices to remark that:

$$E\left[T_c^{(0)}\right] = E\left[T^{\theta}_{-c,c}\right] = E\left[A_{T^{\gamma}_{-c,c}}\right] = E\left[|\sinh\left(B_{T^{\gamma}_{-c,c}}\right)|^2\right].$$

On the one hand, by using the density of $B_{T^{\gamma}_{-c,c}}$:

$$\begin{aligned} E\left[|\sinh\left(B_{T^{\gamma}_{-c,c}}\right)|^4\right] &= \int_{-\infty}^{\infty}\frac{dy}{2c}\,\frac{1}{\cosh(\frac{\pi y}{2c})}|\sinh(y)|^4 \\ &= \int_0^{\infty}\frac{dy}{c}\,\frac{1}{\cosh(\frac{\pi y}{2c})}(\sinh y)^4 \\ &\overset{z=\frac{y}{c}}{=} \int_0^{\infty} dz\,\frac{1}{\cosh(\frac{\pi z}{2})}(\sinh(cz))^4, \end{aligned}$$

which is finite if and only if $c < \frac{\pi}{8}$. In order to prove this, it suffices to use the standard expressions: $\sinh(x) = \frac{e^x - e^{-x}}{2}$ and $\cosh(x) = \frac{e^x+e^{-x}}{2}$. On the other hand (note $T \equiv T^{\gamma}_{-c,c}$), we remark that $-B_T \overset{(law)}{=} B_T$ and [ReY99], ex.3.10, $E\left[e^{kB_T}\right] = E\left[e^{\frac{k^2}{2}T}\right] = \frac{1}{\cos(kc)}$, for $0 \le k < \pi(2c)^{-1}$, thus:

$$\begin{aligned} E\left[|\sinh(B_T)|^4\right] &= \frac{1}{2^4}E\left[\left(e^{B_T} - e^{-B_T}\right)^4\right] \\ &= \frac{1}{2^4}E\left[e^{4B_T} - 4e^{3B_T - B_T} + 6e^{2B_T - 2B_T} - 4e^{B_T - 3B_T} + e^{-4B_T}\right] \\ &= \frac{1}{2^4}\left(2E\left[e^{4B_T}\right] - 8E\left[e^{2B_T}\right] + 6\right) \\ &= \frac{1}{2^3}\left(\frac{1}{\cos(4c)} - 4\frac{1}{\cos(2c)} + 3\right), \end{aligned}$$

which is precisely (2.70) and this is finite if and only if $c < \frac{\pi}{8}$. Moreover, asymptotically for $c \to 0$, by using the scaling property, we have:

$$\begin{aligned} E\left[\left|\sinh\left(B_{T^{\gamma}_{-c,c}}\right)\right|^4\right] &= E\left[\left(\sinh\left(cB_{T^{\gamma}_{-1,1}}\right)\right)^4\right] \overset{c\to 0}{\simeq} c^4 E\left[\left(B_{T^{\gamma}_{-1,1}}\right)^4\right] \\ &= c^4 3 \underbrace{E\left[\left(T^{\gamma}_{-1,1}\right)^2\right]}_{5/3} = 5c^4, \end{aligned}$$

where $E\left[\left(T^{\gamma}_{-1,1}\right)^2\right] = 5/3$ [PiY03] (by using the notation of this paper, Table 3: $E\left[X_t^2\right] = \frac{t(2+3t)}{3}$ for $X_t = C_1$ and $t = 1$). This asymptotics may also be obtained by (2.70) by developing $\cos(4c)$ and $\cos(2c)$ into series up to the second order term and keeping the terms of the order c^4.

□

Remark 2.2.4 *If we slightly modify the above study for the Ornstein-Uhlenbeck process by inserting a diffusion coefficient D:*

$$Z_t = z_0 + \sqrt{2D}\tilde{Z}_t - \lambda \int_0^t Z_s ds \, ,$$

we obtain:

$$\begin{aligned} Z_t &= e^{-\lambda t}\left(z_0 + \sqrt{2D}\int_0^t e^{\lambda s} d\tilde{Z}_s\right) \\ &= e^{-\lambda t}\left(\mathbb{B}_{\alpha_t}\right), \end{aligned} \tag{2.72}$$

where in the second equation we used Dambis-Dubins-Schwarz Theorem with

$$\alpha_t = 2D \int_0^t e^{2\lambda s} ds = D \frac{e^{2\lambda t} - 1}{\lambda}$$

$$\Rightarrow \alpha_t^{-1} = \frac{1}{2\lambda} \ln\left(1 + \frac{\lambda}{D} t\right).$$

So:

$$2\lambda \, E\left[T_c^{(\lambda)}\right] - \ln\left(\frac{\lambda}{D}\right) \overset{\lambda\to\infty}{\longrightarrow} E\left[\ln\left(T^{\theta}_{-c,c}\right)\right], \tag{2.73}$$

because:

$$\begin{aligned} E\left[T_c^{(\lambda)}\right] &= \frac{1}{2\lambda}E\left[\ln\left(1+\frac{\lambda}{D}T^{\theta}_{-c,c}\right)\right] \\ &= \frac{1}{2\lambda}E\left[\ln\left(\frac{\lambda}{D}\left(T^{\theta}_{-c,c}+\frac{D}{\lambda}\right)\right)\right] \\ &= \frac{\ln\left(\frac{\lambda}{D}\right)}{2\lambda}+\frac{1}{2\lambda}E\left[\ln\left(T^{\theta}_{-c,c}+\frac{D}{\lambda}\right)\right]. \end{aligned}$$

Moreover:

$$\begin{aligned} E\left[\ln\left(T^{\theta}_{-c,c}\right)\right] &= 2\ln(z_0)+E\left[\ln\left(T^{\theta(1)}_{-c,c}\right)\right] \\ &= 2\ln(z_0)+\int_0^{\infty}\frac{dz}{\cosh\left(\frac{\pi z}{2}\right)}\ln\left(\sinh\left(cz\right)\right)+\ln\left(2\right)+c_E, \end{aligned} \tag{2.74}$$

where $T^{\theta(1)}_{-c,c}$ denotes the first hitting time of the symmetric conic boundary of angle c for a Brownian motion Z starting from 1.
For λ small, we replace $2T^{\theta}_{-c,c}$ by $\frac{z_0^2}{D}T^{\theta}_{-c,c}$ in the proof of Proposition 2.2.3 (λ small case) and we have:

$$T_c^{(\lambda)} = \frac{1}{2\lambda}\ln\left(1+\lambda\frac{z_0^2}{D}T^{\theta}_{-c,c}\right).$$

By repeating the previous calculation, we make the elementary computation:

$$\frac{1}{\lambda}\left(\frac{\ln\left(1+\frac{z_0^2}{D}x\right)}{2\lambda}-\frac{z_0^2}{D}x\right) = -\frac{1}{2}\int_0^x\frac{\left(\frac{z_0^2}{D}\right)^2 b\,db}{1+\lambda\frac{z_0^2}{D}b}\overset{\lambda\to 0}{\longrightarrow}-\left(\frac{z_0^2}{2D}\right)^2x^2.$$

We replace $x=T^{\theta}_{-c,c}$, and by the dominated convergence theorem [Bil78], for $c<\frac{\pi}{8}$, we obtain:

$$\begin{aligned} \frac{1}{\lambda}\left(E\left[T_c^{(\lambda)}\right]-\frac{z_0^2}{2D}E\left[|\sinh\left(B_{T^{\gamma}_{-c,c}}\right)|^2\right]\right) &\overset{\lambda\to 0}{\longrightarrow} -\frac{1}{3}\left(\frac{z_0^2}{2D}\right)^2E\left[(T^{\theta}_{-c,c})^2\right] \\ &= -\frac{1}{3}\left(\frac{z_0^2}{2D}\right)^2E\left[|\sinh\left(B_{T^{\gamma}_{-c,c}}\right)|^4\right], \end{aligned}$$

where $E\left[|\sinh\left(B_{T^{\gamma}_{-c,c}}\right)|^4\right]$ is given by (2.69), (2.70) and asymptotically, for $c\to 0$ by (2.71).

2.3 On some infinite divisibility properties related with Bougerol's identity

2.3.1 Introduction

a) We take up again Bougerol's celebrated identity in law [Bou83]:

$$\text{for fixed } t > 0, \quad \sinh(B_t) \overset{(law)}{=} \beta_{A_t(B)} \ , \tag{2.75}$$

where$(B_t, t \geq 0)$ is 1-dimensional Brownian motion, $A_t(B) = \int_0^t ds \exp(2B_s)$, and $(\beta_u, u \geq 0)$ is a BM, independent from B. which has recently been the subject of a number of studies and variants (e.g. the previous chapter of this PhD thesis, [Vak10] or [DuY11]).

b) Throughout this section, the inverse functions $(a_s(x),\ x \in \mathbb{R})$, and $(a_c(y),\ y \geq 1)$, denoting:

$$\begin{cases} a_s(x) \equiv \arg\sinh(x) \equiv \log(x + \sqrt{1+x^2}),\ x \in \mathbb{R} \\ a_c(y) \equiv \arg\cosh(y) \equiv \log(y + \sqrt{y^2-1}),\ y \geq 1, \end{cases} \tag{2.76}$$

shall play important roles.
In particular, these two functions satisfy the identity:

$$a_c(1+x) = 2a_s\left(\sqrt{\frac{x}{2}}\right). \tag{2.77}$$

c) A stochastic process $X = (X_t, t \geq 0)$ is called *Lévy process* if $X_0 = 0$ a.s., it has stationary and independent increments and it is almost surely right continuous with left limits. A Lévy process which is increasing is called *subordinator*.

d) Following e.g. [ReY99], a probability measure μ on $\mathbb{R}$ (resp. a real-valued random variable X with law μ) is said to be *infinitely divisible* if, for any $n \geq 1$, there is a probability measure μ_n such that $\mu = \mu_n^{*n}$ (resp. if $X_1, \ldots, X_n$ are n random variables i.i.d., $X \overset{(law)}{=} X_1 + \ldots + X_n$). For instance, Gaussian, Poisson and Cauchy variables are infinitely divisible.

It is proved that (e.g.[ReY99]), μ is infinitely divisible if and only if, its Fourier transform $\hat{\mu}$ is equal to $\exp(\psi)$, with:

$$\psi(u) = ibu - \frac{\sigma^2 u^2}{2} + \int \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) \nu(dx),$$

where $b \in \mathbb{R}$, $\sigma \geq 0$ and ν is a Radon measure on $\mathbb{R} - 0$ such that:

$$\int \frac{x^2}{1+x^2} \nu(dx) < \infty.$$

This formula is known as the *Lévy-Khintchine formula* and the measure ν as the *Lévy measure*.

e) Following [Bon92] (p.29) and [JRY08], a positive random variable Γ is a *generalized Gamma convolution* (GGC) if there exists a positive Radon measure μ on $]0,\infty[$ such that:

$$\begin{aligned} E\left[e^{-\lambda\Gamma}\right] &= \exp\left(- \int_0^\infty \left(1 - e^{-\lambda x}\right) \frac{dx}{x} \int_0^\infty e^{-xz} \mu(dz) \right) &(2.78)\\ &= \exp\left(- \int_0^\infty \log\left(1 + \frac{\lambda}{z}\right) \mu(dz) \right), &(2.79) \end{aligned}$$

with:

$$\int_{]0,1]} |\log x| \mu(dx) \quad \text{and} \quad \int_{[1,\infty[} \frac{\mu(dx)}{x} < \infty. \qquad (2.80)$$

The measure μ is called *Thorin's measure* associated with Γ.

f) In this section, we shall show that $1/A_t(B)$, considered after a suitable measure change from Wiener measure, is infinitely divisible.

2.3.2 Subordinators and an infinite divisibility property

The following statement is taken from, e.g. [JRY08]:

Theorem 2.3.1 *(i) There exist two subordinators* $\left(\mathcal{J}_u^{(0)}, u \geq 0\right)$ *and* $\left(\mathcal{K}_u^{(0)}, u \geq 0\right)$ *with respective Bernstein functions:*

$$i(x) = a_c\,(1+x) \quad \text{and} \quad k(x) = \frac{1}{2}\left(a_c\,(1+x)\right)^2,$$

that is:

$$E\left[\exp\left(-x\ \mathcal{J}_u^{(0)}\right)\right] = \exp\left(-u\ i(x)\right)$$
$$E\left[\exp\left(-x\ \mathcal{K}_u^{(0)}\right)\right] = \exp\left(-u\ k(x)\right).$$

(ii) *The corresponding Lévy measures* $\nu_{\mathcal{J}^{(0)}}(dy)$ *and* $\nu_{\mathcal{K}^{(0)}}(dy)$ *are given by:*

$$\nu_{\mathcal{J}^{(0)}}(y) = dy\ I_0(y)\frac{e^{-y}}{y}; \quad \nu_{\mathcal{K}^{(0)}}(y) = dy\ K_0(y)\frac{e^{-y}}{y}, \tag{2.81}$$

where I_0 *and* K_0 *denote the classical modified Bessel functions, with parameter 0 [Leb72].*

(iii) *The laws of* $\mathcal{J}_u^{(0)}$ *and* $\mathcal{K}_u^{(0)}$*, for any* u*, are generalized Gamma convolutions (GGC) [Bon90] since the functions* $I_0(y)\ e^{-y}$ *and* $K_0(y)\ e^{-y}$ *found in (2.81) are Laplace transforms of the (Thorin) measures* $\mu_{\mathcal{J}^{(0)}}$ *and* $\mu_{\mathcal{K}^{(0)}}$ *[Tho77], i.e.:*

$$I_0(y)\ e^{-y} = \frac{1}{\pi}\int_0^2 e^{-yt}\frac{dt}{\sqrt{t\,(2-t)}}$$

and:

$$K_0(y)\ e^{-y} = \int_2^\infty e^{-yt}\frac{dt}{\sqrt{t\,(t-2)}}.$$

In other terms, we may write:

$$I_0(y)\ e^{-y} = E\left[\exp\left(-2y\ g\right)\right]$$
$$K_0(y)\ e^{-y} = E\left[\exp\left(-2y/g\right)\frac{1}{\sqrt{g}}\right],$$

where g *denotes an arcsine variable, i.e.:*

$$P\left(g \in dt\right) \equiv \frac{dt\ 1_{(0<t<1)}}{\pi\sqrt{t\,(1-t)}}\ .$$

Remark 2.3.2 *In [JRY08], we find more results concerning the GGC variables, coming e.g. from [D-MMY02, MNY02, BFRY06],etc.*

We now come back to Bougerol's identity, which may be written in an equivalent manner to (2.75) as follows:

$$E\left[\frac{1}{\sqrt{2\pi A_t(B)}}\exp\left(-\frac{x^2}{2A_t(B)}\right)\right]=\frac{1}{\sqrt{2\pi t}}\,\frac{1}{\sqrt{1+x^2}}\,\exp\left(-\frac{(a_s(x))^2}{2t}\right)\,, \tag{2.82}$$

or, again equivalently:

$$E_t\left[\exp\left(-\frac{x}{2A_t(B)}\right)\right]=\frac{1}{\sqrt{1+x}}\,\exp\left(-\frac{(a_s(\sqrt{x}))^2}{2t}\right), \tag{2.83}$$

where E_t denotes expectation with respect to $\left(\frac{t}{A_t(B)}\right)^{1/2}\cdot W$, and W is the Wiener measure.

Now, using (2.77), we may also write:

$$\begin{aligned}E_t\left[\exp\left(-\frac{x}{4A_t(B)}\right)\right] &= \frac{1}{\sqrt{1+\frac{x}{2}}}\,\exp\left(-\frac{\frac{1}{4}(a_c(1+x))^2}{2t}\right)\\ &\equiv \frac{1}{\sqrt{1+\frac{x}{2}}}\,\exp\left(-\frac{\frac{1}{2}(a_c(1+x))^2}{4t}\right).\end{aligned} \tag{2.84}$$

Recalling (from Theorem 2.3.1) that $k(x)=\frac{1}{2}\left(a_c\left(1+x\right)\right)^2$, we then obtain easily, from (2.84), that:

$$\frac{1}{4A_t(B)}\,(\text{under } P_t)\overset{(law)}{=}\frac{N^2}{4}+\mathcal{K}^{(0)}_{1/4t}, \tag{2.85}$$

where, on the right hand side of (2.85), the two random objects are independent, with N denoting a standard reduced Gaussian variable.
It was also shown [JRY08] that[§]:

$$\mathcal{K}^{(0)}_u\overset{(law)}{=}\frac{1}{A\left(b_{1/u}\right)}, \tag{2.86}$$

where $b_s\equiv(b_s(v),v\leq s)$ denotes the Brownian bridge of duration s.

Plugging (2.86) to (2.85), we note the remarkable identity:

$$\frac{1}{4A_t(B)}\,(\text{under } P_t)\overset{(law)}{=}\frac{N^2}{4}+\frac{1}{A\left(b_{4t}\right)}. \tag{2.87}$$

[§]If $(f(u),u\leq T)$ is a given function, we denote $A(f)\equiv\int_0^T du\;\exp\left(2f(u)\right)$.

Remark 2.3.3 *In the Appendix A.2, we proceed to some analytic calculations where we show that* $\left(L(x) \equiv \log(\sqrt{x} + \sqrt{1+x}) \equiv a_s(\sqrt{x}),\ x \geq 0\right)$ *and* $\left(\Phi(x) \equiv (\log(\sqrt{x} + \sqrt{1+x}))^2 \equiv (a_s(\sqrt{x}))^2,\ x \geq 0\right)$ *are Bernstein functions.*

Remark 2.3.4 *One can ask whether we can find an infinite divisibility property of* $1/A_t(B)$ *under the Wiener measure* W. *A possible approach may be through the Laplace transforms given in Corollary 2.1.3.*

Chapter 3

The Mean First Rotation Time of a planar polymer

3.1 Stochastic Modeling of a planar polymer

As we have already announced in the introduction, we consider here an approximation where we model a polymer in the plane as a collection of n rigid rods, with equal fixed lengths l_0 and we denote their extremities by $(X_0, X_1, X_2, \ldots, X_n)$ (see Figure 1.3) in a framework with the origin $\mathbf{0}$. We shall fix one of the extreme ends $X_0 = (L, 0)$ (where $L > 0$) on the x-axis.

We repeat that the dynamics of the i-th rod is characterized by its angle $\theta_i(t)$ with respect to the x-axis. The overall polymer dynamics is thus characterized by $(\theta_1(t), \theta_2(t), \ldots, \theta_n(t), t \geq 0)$. We shall consider that due to collisions in the medium, each angle follows a Brownian motion. Thus,

$$(\theta_i(t), i \leq n) \overset{(law)}{=} \sqrt{2D}\,(B_i(t), i \leq n) \Leftrightarrow \begin{cases} d\theta_1(t) = \sqrt{2D}\,dB_1(t) \\ \\ d\theta_2(t) = \sqrt{2D}\,dB_2(t) \\ \\ \vdots \\ d\theta_n(t) = \sqrt{2D}\,dB_n(t), \end{cases} \tag{3.1}$$

where D is the rotational diffusion constant and $(B_1(t), \ldots, B_n(t), t \geq 0)$ is an n-dimensional Brownian motion (BM). The position of each rod can now be obtained as:

$$\begin{cases} X_1(t) = L + l_0 e^{i\theta_1(t)} \\ \\ X_2(t) = X_1(t) + l_0 e^{i\theta_2(t)} \\ \\ \vdots \\ X_n(t) = X_{n-1}(t) + l_0 e^{i\theta_n(t)}. \end{cases} \tag{3.2}$$

In particular, the moving end is given by:

$$X_n(t) = L + l_0 \left(e^{i\theta_1(t)} + \ldots + e^{i\theta_n(t)} \right) = L + l_0 \sum_{k=1}^{n} e^{i\theta_k(t)}, \tag{3.3}$$

which can be written as:

$$X_n(t) = R_n(t) e^{i\varphi_n(t)}, \tag{3.4}$$

and thus $\varphi_n(t)$ defines the rotation of the polymer with respect to the origin $\mathbf{0}$ and R_n is the associated distance (to the origin).

In order to compute the MRT, we shall study a sum of exponentials of Brownian motions, a topic which often leads to surprising computations [Yor01]. First, we scale the space and time variables as follows:

$$\tilde{l} = \frac{L}{l_0} \quad \text{and} \quad \tilde{t} = \frac{t}{2D}. \tag{3.5}$$

Equation (3.3) becomes:

$$X_n(t) = \tilde{l} + \sum_{k=1}^{n} e^{i\tilde{B}_k(t)}, \tag{3.6}$$

where $\left(\tilde{B}_1(t), \ldots, \tilde{B}_n(t), t \geq 0 \right)$ is an n-dimensional Brownian motion (BM) and for $k = 1, \ldots, n$, using the scaling property of Brownian motion, we have:

$$\tilde{B}_k(t) \equiv \frac{1}{\sqrt{2D}} B_k(t) \overset{(law)}{=} B_k\left(\frac{t}{2D} \right) = B_k(\tilde{t}).$$

Before describing our approach, we first discuss the mean initial configuration of the polymer. It is given by:

$$c_n = E\left(\sum_{k=1}^{n} e^{i\theta_k(0)}\right), \tag{3.7}$$

where the initial angles $\theta_k(0)$ are such that the polymer has not already made a loop. After scaling, the mean initial configuration becomes:

$$c_n = E\left(\sum_{k=1}^{n} e^{i\,\tilde{\theta}_k(0)}\right), \tag{3.8}$$

with:

$$\tilde{\theta}_k(0) = \tilde{B}_k(0). \tag{3.9}$$

From now on, we shall use θ_k instead of $\tilde{\theta}_k$, B instead of $\tilde{B}$ and t instead of $\tilde{t}$. Any segment in the interior of the polymer can hit the angle 2π around the origin, but we will not consider this as a winding event, although we could and in that case, the MRT would be different ([AKH11] for some examples). Rather, we shall only consider that given an initial configuration c_n, the MRT is defined as:

$$MRT \equiv E\{\tau_n | c_n\} \equiv E\left[\tau_n\right], \tag{3.10}$$

where:

$$\tau_n \equiv \inf\{t > 0, |\varphi_n(t)| = 2\pi\}. \tag{3.11}$$

Thus, an initial configuration is not winding when:

$$|\varphi_n(0)| < 2\pi. \tag{3.12}$$

Thus, we can define the winding event using only a one dimensional variable. In general, winding is a rare event and we expect that the MRT will depend crucially on the length of the polymer which will be quite long. Interestingly, the rotation is accomplished when the angle $\varphi_n(t)$ reaches 2π or -2π, but the distance of the free end point to the origin is not fixed, leading to a one dimensional free parameter space. This undefined position is in favor of a winding time that is not too large compared to any narrow escape problem where a Brownian particle has to find a small target in a confined domain [WaK93, HoS04, SSHE06, SSH07, BKH07, PWPSK09].

In this study, we will consider not only that the initial condition satisfies $|\varphi_n(0)| < 2\pi$, but we shall impose that $\varphi_n(0)$ is located far enough from 2π, to avoid studying any boundary layer effect, which would lead to a different MRT law. Indeed, starting inside the boundary layer for a narrow escape type problem leads to specific escape laws [SSHE06]. Given a small $\varepsilon > 0$ we shall consider the space of configurations Ω_ε such that $|\varphi_n(0)| < 2\pi - \varepsilon$. We shall mainly focus on the stretched polymer:

$$(\theta_1(0), \theta_2(0), \ldots, \theta_n(0)) = (0, 0, \ldots, 0) \tag{3.13}$$

and thus $c_n = n > 0$ (in this case, $\varphi_n(0) = 0$). Finally, it is quite obvious that winding occurs only when the condition:

$$nl_0 > L \tag{3.14}$$

is satisfied, which we assume all along.

The outline of our method is: first we show that the sum $X_n(t)$ converges (eq. (3.3)) and we obtain a Central Limit Theorem. Using Itô calculus, we study the sequence:

$$\frac{1}{\sqrt{n}}\overline{X}_n(t) = \frac{1}{\sqrt{n}}\left[X_n(t) - E\left(X_n(t)\right)\right]. \tag{3.15}$$

and we prove that $\frac{1}{\sqrt{n}}\overline{X}_n(t)$, for n large, converges to a stochastic process which is a generalization of an Ornstein-Uhlenbeck process (GOUP), containing a time dependent deterministic drift $c_n e^{-t}$. This GOUP is driven by a martingale $\left(M_t^{(n)}, t \geq 0\right)$ that we further characterize. Interestingly, the two cartesian coordinates of $\left(M_t^{(n)}, t \geq 0\right)$ converge to two independent Brownian motions with two different time scale functions. To obtain an asymptotic formula for the MRT, we show that in the long time asymptotics, where winding occurs, the GOUP can be approximated by a standard Ornstein-Uhlenbeck process (OUP). Using some properties of the GOUP [Vak10], we finally derive the MRT for the polymer which is the mean time that $|\varphi_n(t)| = 2\pi$.

3.2 Properties of the free polymer end $X_n(t)$ using a Central Limit Theorem

In this section we study some properties of the free polymer end $X_n(t)$. In particular, using a Central Limit Theorem, we show that the limit process satisfies a stochastic equation of a new type. To study the random part of $X_n(t)$, we shall remove from it its first moment and we shall now consider the asymptotic behavior of the drift-less sequence:

$$\frac{1}{\sqrt{n}}\overline{X}_n(t) = \frac{1}{\sqrt{n}}\left[X_n(t) - E\left(X_n(t)\right)\right]. \tag{3.16}$$

We start by computing the first moment $E\left(X_n(t)\right)$. Because in relation (3.3), $(\theta_i(t), t \geq 0)$ are assumed to be n independent identically distributed (iid) Brownian motions with variance $2D$, after rescaling, we obtain:

$$\begin{aligned} E\left(X_n(t)\right) &= E\left[\tilde{l} + \sum_{k=1}^{n} e^{iB_k(t)}\right] \\ &= \tilde{l} + \left(\sum_{k=1}^{n} E\left[e^{i(B_k(t)-B_k(0))}\right] \; E\left[e^{i(B_k(0))}\right]\right) \\ &= \tilde{l} + c_n e^{-\frac{t}{2}}, \end{aligned} \tag{3.17}$$

where c_n is defined by (3.8) and we have used that:

$$E\left[e^{i(B_k(t)-B_k(0))}\right] = e^{-\frac{t}{2}} \; . \tag{3.18}$$

We study the sequence (3.16) as follows:

$$\begin{aligned} \frac{1}{\sqrt{n}}\overline{X}_n(t) &= \frac{1}{\sqrt{n}}\left[\sum_{k=1}^{n} e^{iB_k(t)} - E\left(\sum_{k=1}^{n} e^{iB_k(t)}\right)\right] \\ &= \frac{1}{\sqrt{n}}\sum_{k=1}^{n} F_k(t), \end{aligned} \tag{3.19}$$

where $F_k(t) = e^{iB_k(t)} - E\left(e^{iB_k(t)}\right)$. Applying Itô's formula to:

$$Z_t^{(n)} = \frac{1}{\sqrt{n}}\sum_{k=1}^{n} F_k(t), \tag{3.20}$$

with $Z_0^{(n)} = 0$, we obtain:

$$\begin{aligned} Z_t^{(n)} &= \frac{i}{\sqrt{n}} \int_0^t \sum_{k=1}^n e^{iB_k(s)} dB_k(s) - \frac{1}{2\sqrt{n}} \int_0^t \sum_{k=1}^n \left(e^{iB_k(s)} - E\left(e^{iB_k(s)}\right)\right) ds = \\ &= M_t^{(n)} - \frac{1}{2}\int_0^t Z_s^{(n)}\, ds, \end{aligned} \tag{3.21}$$

where:

$$\begin{aligned} M_t^{(n)} &= \frac{1}{\sqrt{n}} i \int_0^t \sum_{k=1}^n e^{iB_k(s)}\, dB_k(s) \\ &= \frac{1}{\sqrt{n}} \int_0^t \sum_{k=1}^n \left(i\cos(B_k(s)) - \sin(B_k(s))\right)\, dB_k(s) \\ &= -S_t^{(n)} + iC_t^{(n)}. \end{aligned} \tag{3.22}$$

We shall now study the asymptotic limit of the martingales $M_t^{(n)}$ as $n \to \infty$ and summarize our result in the following theorem.

Theorem 3.2.1 *The sequence $(M_t^{(n)}, t \geq 0)$ converges in law to a 2-dimensional Brownian motion, viewed at a nonlinear function of time. We obtain a convergence in law associated with the topology of the uniform convergence on compact sets of the functions in $C(\mathbb{R}_+, \mathbb{R}^2)$. More precisely,*

$$(S_t^{(n)}, C_t^{(n)}, t \geq 0) \xrightarrow[n\to\infty]{(law)} (\hat{\tilde{\beta}}_{\left(\frac{1}{2}\int_0^t ds\,(1-e^{-2s})\right)}, \tilde{\beta}_{\left(\frac{1}{2}\int_0^t ds\,(1+e^{-2s})\right)}, t \geq 0), \tag{3.23}$$

where $(\hat{\tilde{\beta}}_u, \tilde{\beta}_u, u \geq 0)$ are two independent Brownian motions.

Remark 3.2.2 *The convergence in law (3.23) is a new result and it shows that the sum of the complex exponentials of i.i.d. Brownian motions can be approximated by a two dimensional Brownian motion, with a different time scale for each coordinate.*

Proof of Theorem 3.2.1 See Appendix A.3.

□

We conclude that the sequence $(M_t^{(n)}, t \geq 0)$ converges in law:

$$(M_t^{(n)}, t \geq 0) \underset{n\to\infty}{\overset{(law)}{\longrightarrow}} (\hat{\tilde{\beta}}_{\left(\frac{t}{2}-\frac{1-e^{-2t}}{4}\right)} + i\tilde{\beta}_{\left(\frac{t}{2}+\frac{1-e^{-2t}}{4}\right)}, t \geq 0). \tag{3.24}$$

The process $(Z_t^{(n)}, t \geq 0)$, which is defined (3.21), is a generalization of the classical Ornstein-Uhlenbeck process; it is driven by $(M_t^{(n)}, t \geq 0)$ and, from (3.21), we obtain:

$$Z_t^{(n)} = e^{-\frac{t}{2}} \int_0^t e^{\frac{s}{2}} dM_s^{(n)}. \tag{3.25}$$

Corollary 3.2.3 a) *The sequence* $(Z_t^{(n)}, t \geq 0)$ *converges in law:*

$$Z_t^{(n)} \underset{n\to\infty}{\overset{(law)}{\longrightarrow}} Z_t^{(\infty)}, \tag{3.26}$$

with:

$$Z_t^{(\infty)} = e^{-\frac{t}{2}} \int_0^t \sqrt{\sinh(s)}\, d\delta_s + i\, e^{-\frac{t}{2}} \int_0^t \sqrt{\cosh(s)}\, d\tilde{\delta}_s, \tag{3.27}$$

where $\left(\delta_t, \tilde{\delta}_t, t \geq 0\right)$ *are two independent 1-dimensional Brownian motions.*
b) $Z_t^{(\infty)} \underset{t\to\infty}{\overset{(law)}{\longrightarrow}} \frac{1}{\sqrt{2}}\left(N + i\tilde{N}\right)$, *where* N *and* $\tilde{N}$ *are two centered and reduced Gaussian variables.*

Proof of Corollary 3.2.3
a) From (3.22), (3.23) and (3.25) we deduce:

$$Z_t^{(n)} = e^{-\frac{t}{2}} \int_0^t e^{\frac{s}{2}} dM_s^{(n)} \tag{3.28}$$

$$\underset{n\to\infty}{\overset{(law)}{\longrightarrow}} e^{-\frac{t}{2}} \int_0^t \underbrace{e^{\frac{s}{2}} \sqrt{\frac{1-e^{-2s}}{2}}}_{\sqrt{\sinh(s)}}\, d\delta_s + ie^{-\frac{t}{2}} \int_0^t \underbrace{e^{\frac{s}{2}} \sqrt{\frac{1+e^{-2s}}{2}}}_{\sqrt{\cosh(s)}}\, d\tilde{\delta}_s \equiv Z_t^{(\infty)}. \tag{3.29}$$

b) From (3.29), we change variables: $u = t - s$ and we obtain:

$$\begin{aligned} Z_t^{(\infty)} &= \int_0^t e^{-\frac{t-s}{2}} \sqrt{\frac{1-e^{-2s}}{2}}\, d\delta_s + i \int_0^t e^{-\frac{t-s}{2}} \sqrt{\frac{1+e^{-2s}}{2}}\, d\tilde{\delta}_s \\ &\underset{(law)}{\overset{u=t-s}{=}} \int_0^t e^{-\frac{u}{2}} \sqrt{\frac{1}{2} - \frac{e^{-2(t-u)}}{2}}\, d\delta_u + i \int_0^t e^{-\frac{u}{2}} \sqrt{\frac{1}{2} + \frac{e^{-2(t-u)}}{2}}\, d\tilde{\delta}_u \end{aligned} \tag{3.30}$$

$$\xrightarrow[t\to\infty]{L^2} \frac{1}{\sqrt{2}} \int_0^\infty e^{-\frac{u}{2}} d\delta_u + i\, \frac{1}{\sqrt{2}} \int_0^\infty e^{-\frac{u}{2}} d\tilde{\delta}_u\ , \tag{3.31}$$

where the two variables on the RHS of (3.31) are centered Gaussian with variance $1/2$ and the convergence in L^2 for $t \to \infty$ may be proved by using the dominated convergence theorem.

□

Remark 3.2.4 *We should mention that a similar study concerning the limit process $Z_t(\varphi) = \lim_{n\to\infty} Z_t^{(n)}(\varphi)$ has already been done [Itô83] for the class of functions φ which are C^∞ of compact support. So, this work could be regarded as a variant for the function $\varphi(x) = e^{i\lambda x}$, $\lambda > 0$. More precisely, K. Itô in this article studies the limit process $X_t = \lim_{n\to\infty} X_n(t)$ with:*

$$X_n(t) = X_n(t, \varphi) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \left(\varphi(B_k(t)) - E[\varphi(B_k(t))]\right), \tag{3.32}$$

where $(B_k(t), k = 1, 2, 3, \ldots)$ is a sequence of independent 1-dimensional Brownian motions with a common initial distribution μ and φ are C^∞ of compact support. He proves that X_t exists and moreover it satisfies a Stochastic Differential Equation.

$$dX_t = (\partial \circ \sqrt{\mu_t})db_t + \frac{1}{2}\partial^2 X_t dt\ , \tag{3.33}$$

where $\{b_t\}$ is a standard Wiener $\mathcal{J}_2'$ process [Itô83] paragraph 4, $\mu_t = \mu * g_t$ (g_t: the Gauss density with mean 0 and variance t), $\sqrt{\mu_t}(\in C^\infty(\mathbb{R}))$ is a multiplication operator in $\mathcal{D}'$, ∂ is the differentiation in $\mathcal{D}'$ and $\partial \circ \sqrt{\mu_t}$ is the composition of these operators.

Moreover, this study could be connected to some more results [Ochi85]. In this article, a limit theorem is proved (by using some methods similar to the one used in this PhD thesis) concerning the semimartingale $X = (X_t(\cdot))_{t\geq 0}$ and more precisely concerning $X^\lambda = (\lambda^{-1/2} X_{\lambda t}(\cdot))_{t\geq 0}$.

On the one hand, from the identities (3.16), (3.17), (3.19) and (3.20), we obtain the following expansion:

$$\begin{aligned} X_n(t) &= \overline{X}_n(t) + E\left[X_n(t)\right] \\ &= \sqrt{n}\, Z_t^{(n)} + c_n e^{-\frac{t}{2}} + \tilde{l} \\ \Rightarrow \tilde{X}_n(t) &\equiv X_n(t) - \tilde{l} = \sqrt{n}\, Z_t^{(n)} + c_n e^{-\frac{t}{2}}, \end{aligned} \tag{3.34}$$

with n the number of rods/beads, $Z_t^{(n)}$ a GOUP driven by $M_t^{(n)}$ which is given by (3.24), $c_n = E\left(\sum_{k=1}^{n} e^{i\theta_k(0)}\right)$ a constant depending on the mean initial configuration and $\tilde{l} = \frac{L}{l_0}$ the rescaled distance of the fixed end from the origin $\mathbf{0}$ (l_0 is the fixed length of the rods).
On the other hand, with Corollary 3.2.3(a), we deduce that:

$$\begin{aligned} Z_t^{(n)} &\equiv \frac{1}{\sqrt{n}} \overline{X}_n(t) \equiv \frac{1}{\sqrt{n}} \left[X_n(t) - E\left(X_n(t)\right)\right] \\ &= \frac{1}{\sqrt{n}} \left[X_n(t) - c_n e^{-\frac{t}{2}} - \tilde{l}\right] \\ &\xrightarrow[n\to\infty]{(law)} e^{-\frac{t}{2}} \int_0^t \sqrt{\sinh(s)}\, d\delta_s + i\, e^{-\frac{t}{2}} \int_0^t \sqrt{\cosh(s)}\, d\tilde{\delta}_s \equiv Z_t^{(\infty)}. \end{aligned} \tag{3.35}$$

3.3 Asymptotic expression for the MRT

Estimation of the MRT

We now study more precisely the different time scales of the two Brownian motions in (3.24) and (3.35). We estimate $Z_t^{(\infty)}$ for t large, the regime for which the rotation will be accomplished.
We introduce here the following notation: $\overset{L^2}{\approx}$ denotes closeness in the L^2-norm: for two stochastic processes $(W_t^{(1)}, t \geq 0)$ and $(W_t^{(2)}, t \geq 0)$, the notation $W_t^{(1)} \overset{L^2}{\approx} W_t^{(2)}$ means that $\lim_{t\to\infty} E\left[\left|W_t^{(1)} - W_t^{(2)}\right|^2\right] = 0$.
We shall show that, with $\left(\mathbb{B}_t = \delta_t + i\tilde{\delta}_t, t \geq 0\right)$ a 2-dimensional Brownian

motion starting from 1:

$$\begin{aligned} Z_t^{(\infty)} &= e^{-\frac{t}{2}} \int_0^t \sqrt{\sinh(s)}\, d\delta_s + i\, e^{-\frac{t}{2}} \int_0^t \sqrt{\cosh(s)}\, d\tilde{\delta}_s \\ &\overset{L^2}{\approx} e^{-\frac{t}{2}} \int_0^t \frac{e^{\frac{s}{2}}}{\sqrt{2}}\, d\mathbb{B}_s. \end{aligned} \tag{3.36}$$

For this, it suffices to use the expression (3.31) and the following Proposition, which reinforces the $\overset{L^2}{\approx}$ result in (3.36).

Proposition 3.3.1 *As $t \to \infty$, the Gaussian martingales:*

$$\left(\int_0^t \sqrt{\sinh(s)}\, d\delta_s - \int_0^t \frac{e^{s/2}}{\sqrt{2}}\, d\delta_s, t \geq 0 \right),$$

and

$$\left(\int_0^t \sqrt{\cosh(s)}\, d\tilde{\delta}_s - \int_0^t \frac{e^{s/2}}{\sqrt{2}}\, d\tilde{\delta}_s, t \geq 0 \right)$$

converge a.s. and in L^2. The limit variables are Gaussian with variances $\frac{\pi-3}{2}$ and $-1 + 2\sqrt{2} - 2a_s(1) \approx 0,033$, where $a_s(x) \equiv \arg\sinh(x) \equiv \log(x + \sqrt{1+x^2})$, $x \in \mathbb{R}$, respectively.

Thus, by multiplying both processes by $e^{-\frac{t}{2}}$, we obtain (3.36).

Proof of Proposition 3.3.1 The Gaussian martingale $\int_0^t \left(\sqrt{\sinh(s)} - \frac{e^{s/2}}{\sqrt{2}} \right) d\delta_s$ has increasing process

$$\int_0^t \left(\sqrt{\sinh(s)} - \frac{e^{s/2}}{\sqrt{2}} \right)^2 ds = \int_0^t \frac{e^s}{2} \left(\sqrt{1 - e^{-2s}} - 1 \right)^2 ds\ ,$$

which converges as $t \to \infty$. Hence, the limit variable $\int_0^\infty \left(\sqrt{\sinh(s)} - \frac{e^{s/2}}{\sqrt{2}} \right) d\delta_s$ is Gaussian, and its variance is given by (we change variables: $u = e^{-2s}$ and

$B(a,b)$ denotes the Beta function with arguments a and b**):

$$\begin{aligned} \int_0^\infty \frac{ds\, e^s}{2} \left(\sqrt{1-e^{-2s}}-1\right)^2 &= \frac{1}{4}\int_0^1 du\, u^{-3/2}\left(\sqrt{1-u}-1\right)^2 \\ &= \frac{1}{4}\left[\int_0^1 du\, u^{-3/2}\left((1-u)-2\sqrt{1-u}+1\right)\right] \\ &= \frac{1}{4}\left\{B\left(-\frac{1}{2},2\right)-2B\left(-\frac{1}{2},\frac{3}{2}\right)-2\right\} = \frac{\pi-3}{2}\ . \end{aligned}$$

To be rigorous, the integral $\int_0^1 du\, u^{-\alpha}\left(\sqrt{1-u}-1\right)^2$, which is well defined for $0<\alpha<1$, can be extended analytically for any complex α with $\mathrm{Re}(\alpha)<3$. For the convergence of the second process, it suffices to replace $\sinh(s)$ by $\cosh(s) \equiv \frac{e^s}{2}(1+e^{-2s})$. The limit variable $\int_0^\infty \left(\sqrt{\cosh(s)}-\frac{e^{s/2}}{\sqrt{2}}\right)\, d\tilde{\delta}_s$ is also Gaussian, and, by repeating the previous calculation, we easily compute its variance.

$\square$

Asymptotic expression for the MRT

For t large, we derive an asymptotic value for the MRT. First, from (3.25) and (3.34):

$$\begin{aligned} \tilde{X}_n(t) &= \sqrt{n}Z_t^{(n)} + c_n e^{-\frac{t}{2}} \\ &= \sqrt{n}e^{-\frac{t}{2}}\left(\tilde{c}_n + \int_0^t e^{\frac{s}{2}}dM_s^{(n)}\right), \end{aligned} \tag{3.37}$$

where, the sequence $\tilde{c}_n$ is:

$$\tilde{c}_n \equiv \frac{c_n}{\sqrt{n}}, \tag{3.38}$$

n is the number of rods/beads, and $c_n \equiv E\left(\sum_{k=1}^n e^{i\theta_k(0)}\right)$ is a constant depending on the mean initial configuration. Thus, from (3.34), (3.36), (3.37) and also using the scaling property of Brownian motion:

$$\tilde{X}_n(t) \underset{n:large}{\overset{(law)}{\approx}} \sqrt{n}Y_t^{(n)}, \tag{3.39}$$

**We recall that if $(\Gamma(x), x\geq 0)$ denotes the Gamma function, then $B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

where:

$$Y_t^{(n)} \equiv e^{-\frac{t}{2}} \left(\tilde{c}_n + \int_0^t e^{\frac{s}{2}} d\mathbb{B}_{s/2} \right). \tag{3.40}$$

Changing time and expression of the MRT

To express the MRT, we now apply deterministic time changes. To make our writing simple, we denote Y_t for $Y_t^{(n)}$, and changing variables $u = \frac{s}{2}$ in (3.40), we obtain:

$$Y_{2t} = e^{-t} \left(\tilde{c}_n + \int_0^t e^u d\mathbb{B}_u \right). \tag{3.41}$$

Now, (following section 2.2 for $\lambda = 1$ and $z_0 = \tilde{c}_n$), by Dambis-Dubins-Schwarz Theorem, there is another BM $(\tilde{\mathbb{B}}_t, t \geq 0)$, starting from $\tilde{c}_n$, such that:

$$Y_{2t} = e^{-t} \left(\tilde{\mathbb{B}}_{\alpha_t} \right), \tag{3.42}$$

where

$$\alpha_t = \int_0^t e^{2s} ds = \frac{e^{2t} - 1}{2},$$

hence:

$$\alpha^{-1}(t) = \frac{1}{2} \ln (1 + 2t). \tag{3.43}$$

By applying Itô's formula to (3.42), we obtain:

$$dY_{2s} = -e^{-s} \tilde{\mathbb{B}}_{\alpha_s} ds + e^{-s} d \left(\tilde{\mathbb{B}}_{\alpha_s} \right).$$

We divide by Y_{2s} and we obtain:

$$\frac{dY_{2s}}{Y_{2s}} = -ds + \frac{d\tilde{\mathbb{B}}_{\alpha_s}}{\tilde{\mathbb{B}}_{\alpha_s}}.$$

Thus:

$$\mathrm{Im} \left(\frac{dY_{2s}}{Y_{2s}} \right) = \mathrm{Im} \left(\frac{d\tilde{\mathbb{B}}_{\alpha_s}}{\tilde{\mathbb{B}}_{\alpha_s}} \right),$$

which means that for:

$$\theta_t^Z \equiv \mathrm{Im}(\int_0^t \frac{dZ_s}{Z_s}), t \geq 0, \tag{3.44}$$

denoting the continuous winding process associated to a stochastic process Z, we have:

$$\theta_{2t}^Y = \theta_{\alpha_t}^{\tilde{\mathbb{B}}}.$$

Thus, the first hitting times of the symmetric conic boundary of angle c:

$$T_c^{(\lambda)} \equiv \inf\left\{t \geq 0 : \left|\theta_t^Y\right| = c\right\}, \tag{3.45}$$

and

$$T_{-c,c}^{\theta} \equiv \inf\left\{t \geq 0 : \left|\theta_t^{\tilde{\mathbb{B}}}\right| = c\right\}, \tag{3.46}$$

for the Ornstein-Uhlenbeck process Y with parameter $\lambda \geq 0$ (here, $\lambda = 1$) and for a Brownian motion $\tilde{\mathbb{B}}$ respectively, with relation (3.43), satisfy:

$$2T_c^{(\lambda)} = \frac{1}{2}\ln\left(1 + 2T_{-c,c}^{\theta}\right). \tag{3.47}$$

Finally,

$$\begin{aligned} E\left[2T_c^{(\lambda)}\right] &= \frac{1}{2}E\left[\ln\left(1 + 2T_{-c,c}^{\theta}\right)\right] && (3.48) \\ &= \frac{\ln 2}{2} + \frac{1}{2}E\left[\ln\left(T_{-c,c}^{\theta} + \frac{1}{2}\right)\right], && (3.49) \end{aligned}$$

and equivalently:

$$E\left[T_c^{(\lambda)}\right] = \frac{\ln 2}{4} + \frac{1}{4}E\left[\ln\left(T_{-c,c}^{\theta} + \frac{1}{2}\right)\right]. \tag{3.50}$$

Thus, by taking $c = 2\pi$, for n large, with $\tilde{\varphi}_n(t)$ denoting the total angle of $\tilde{X}_n(t)$, the mean time $E\left[\tilde{\tau}_n\right]$, where $\tilde{\tau}_n \equiv \inf\{t > 0, |\tilde{\varphi}_n(t)| = 2\pi\}$, that $\tilde{X}_n(t)$ rotates around $\mathbf{0}$, is:

$$E\left[\tilde{\tau}_n\right] \approx \frac{\sqrt{n}}{4}\left(\ln 2 + E\left[\ln\left(T_{2\pi}^{|\theta^{\tilde{\mathbb{B}}}|} + \frac{1}{2}\right)\right]\right). \tag{3.51}$$

By using the simple inequality $\log(1+x) \leq x$, we shall now estimate the second term:

$$E\left[\ln\left(T^{\theta}_{-c,c}+\frac{1}{2}\right)\right] - E\left[\ln\left(T^{\theta}_{-c,c}\right)\right] = E\left[\ln\left(1+\frac{1}{2T^{\theta}_{-c,c}}\right)\right] \tag{3.52}$$

$$\leq \frac{1}{2}E\left[\frac{1}{T^{\theta}_{-c,c}}\right]. \tag{3.53}$$

We shall estimate the first moment of $1/T^{\theta}_{-c,c}$. For planar Brownian motion $(\tilde{\mathbb{B}}_t, t \geq 0)$ starting from $\tilde{c}_n + i0, \tilde{c}_n > 0$, from the skew product representation (equations (2.2) and (2.6)) [ReY99] there is another planar Brownian motion $(\zeta_u + i\gamma_u, u \geq 0)$ starting from $\log \tilde{c}_n + i0$, such as:

$$\log\left|\tilde{\mathbb{B}}_t\right| + i\theta_t \equiv \int_0^t \frac{d\tilde{\mathbb{B}}_s}{\tilde{\mathbb{B}}_s} = (\zeta_u + i\gamma_u)\Big|_{u=H_t\equiv\int_0^t \frac{ds}{|\tilde{\mathbb{B}}_s|^2}}, \tag{3.54}$$

and equivalently:

$$\log\left|\tilde{\mathbb{B}}_t\right| = \zeta_{H_t}; \quad \theta_t = \gamma_{H_t}. \tag{3.55}$$

Thus, with $T^{\gamma}_{-c,c} \equiv \inf\{t \geq 0 : |\gamma_t| = c\}$ and $T^{\theta}_{-c,c} \equiv \inf\{t \geq 0 : |\theta_t| = c\}$, because $\theta_{T^{\theta}_{-c,c}} = \gamma_{H_{T^{\theta}_{-c,c}}}$:

$$T^{\gamma}_{-c,c} = H_{T^{\theta}_{-c,c}},$$

hence $T^{\theta}_{-c,c} = H^{-1}_u\Big|_{u=T^{\gamma}_{-c,c}}$, where:

$$H^{-1}_u \equiv \inf\{t : H_t > u\} = \int_0^u ds \exp(2\zeta_s) \equiv A_u, \tag{3.56}$$

and for $u = T^{\gamma}_{-c,c}$, we obtain:

$$T^{\theta}_{-c,c} = A_{T^{\gamma}_{-c,c}}. \tag{3.57}$$

Concerning the moments of $T^{\theta}_{-c,c}$, we have the following (some discussions have already been made, e.g. [MaY05]):

Theorem 3.3.2 *For every $c > 0$, $T^{\theta}_{-c,c}$ enjoys the following integrability properties:*

(i) for $p > 0$, $E\left[\left(T^{\theta}_{-c,c}\right)^p\right] < \infty$, *if and only if* $p < \frac{\pi}{4c}$.

(ii) for any $p < 0$, $E\left[\left(T^{\theta}_{-c,c}\right)^p\right] < \infty$.

Corollary 3.3.3 *For* $0 < c < d$, *the random times*

$$T^{\theta}_{-d,c} \equiv \inf\{t : \theta_t \notin (-d,c)\} \quad \textit{and} \quad T^{\theta}_{c} \equiv \inf\{t : \theta_t \notin (-\infty,c)\},$$

satisfy the inequality:

$$T^{\theta}_{c} \geq T^{\theta}_{-d,c} \geq T^{\theta}_{-c,c}. \tag{3.58}$$

Thus, for $p > 0$, *their negative moments satisfy:*

$$E\left[\frac{1}{(T^{\theta}_{c})^p}\right] \leq E\left[\frac{1}{\left(T^{\theta}_{-d,c}\right)^p}\right] \leq E\left[\frac{1}{\left(T^{\theta}_{-c,c}\right)^p}\right] < \infty. \tag{3.59}$$

Proof of Theorem 3.3.2
(i) The proof is given by Spitzer [Spi58] and others [Bur77], [ReY99] Ex. 2.21/page 196.
(ii) We use the representation $T^{\theta}_{-c,c} = A_{T^{\gamma}_{-c,c}}$ together with a recurrence formula for the negative moments of A_t [Duf00], Theorem 4.2, p. 417 (in fact, Dufresne also considers $A^{(\mu)}_t = \int_0^t ds \exp(2\zeta_s + 2\mu s)$, but we only need to take $\mu = 0$ for our purpose, and we note $A_t \equiv A^{(0)}_t$):

$$\begin{aligned}
E\left[(2A_t)^{-p}\right] &= \int_0^{\infty} \phi(p,t,y)\coth(y)dy, \quad p = 1,2,\dots\,, \\
\phi(1,t,y) &= \frac{ye^{-y^2/2t}}{\sqrt{2\pi t^3}}, \\
\phi(p,t,y) &= \frac{1}{2(1-p)}\frac{\vartheta}{\vartheta t}\phi(p-1,t,y) + (p-1)\phi(p-1,t,y), \quad p = 2,3,\dots\,.
\end{aligned} \tag{3.60}$$

As a consequence of (3.60), we easily obtain that $E\left[\frac{1}{(A_t)^p}\right]$ may be majorized by a linear combination with positive coefficients, of powers of t, either positive or negative. As an example, we give details for $p = 1$:

with K a positive constant,

$$\begin{aligned}
E\left[\frac{1}{A_t}\right] &= 2\int_0^\infty \frac{ye^{-y^2/2t}}{\sqrt{2\pi t^3}} \coth(y)\; dy \\
&\overset{y=z\sqrt{t}}{=} \sqrt{2}\int_0^\infty \frac{z}{\sqrt{\pi t}} e^{-z^2/2} \coth\left(z\sqrt{t}\right)\; dz \\
&= \sqrt{2}\int_0^\infty \frac{z\sqrt{t}}{t\sqrt{\pi}} e^{-z^2/2} \coth\left(z\sqrt{t}\right)\; dz \\
&\leq K\left(\frac{1}{\sqrt{t}} + \frac{1}{t}\right),
\end{aligned}$$

because $z\sqrt{t}\coth\left(z\sqrt{t}\right) \leq K(1 + z\sqrt{t})$.
Coming back to the general case, by replacing $t = T^{\gamma}_{-c,c}$ it suffices to show that:

$$E\left[\left(T^{\gamma}_{-c,c}\right)^q\right] < \infty,$$

for any $q \in \mathbb{R}$.
For $q > 0$, this is clear, i.e. we may invoke Burkholder-Davis-Gundy inequality (BDG) [KaSh88, ReY99].
For $q < 0$, we use:

$$\frac{1}{T^{\gamma}_{-c,c}} \overset{(law)}{=} \frac{1}{c^2}\left(\sup_{s\leq 1}|\gamma_s|\right)^2,$$

a well-known consequence of the scaling property of Brownian motion [ReY99], Prop. 3.10, p.108.
Now, $\left(\sup_{s\leq 1}|\gamma_s|\right)^2$ admits even some exponential moments.

□

Moreover, the first negative moment of A_t has the following integral representation for any $t > 0$ ([D-MMY00], p.49, Prop. 7, formula (15)):

$$E\left[\frac{1}{A_t}\right] = \int_0^\infty ye^{-y^2t/2}\coth\left(\frac{\pi}{2}y\right)\; dy. \tag{3.61}$$

With $\zeta_s = \log(\tilde{c}_n) + \zeta_s^{(0)}$, where $(\zeta_s^{(0)}, s \geq 0)$ is another one-dimensional Brownian motion starting from 0, we obtain:

$$\begin{aligned} T^{\theta}_{-c,c} &= H^{-1}(T^{\gamma}_{-c,c}) \equiv \int_0^{T^{\gamma}_{-c,c}} ds\ \exp\left(2\zeta_s\right) \\ &= (\tilde{c}_n)^2 \left(\int_0^{T^{\gamma}_{-c,c}} ds\ \exp\left(2\zeta_s^{(0)}\right) \right) \\ &\equiv (\tilde{c}_n)^2 T^{\theta^{(1)}}_{-c,c}\ , \end{aligned} \tag{3.62}$$

where $T^{\theta^{(1)}}_{-c,c} \equiv \inf\left\{t \geq 0 : \left|\theta_t^Z\right| = c\right\}$ is the first hitting time of the symmetric conic boundary of angle c of a Brownian motion Z starting from $1 + i0$. Hence, from (3.57), (3.61) and by using the Laplace transform for the hitting time $T^{\gamma}_{-c,c}$ for a reflecting Brownian motion [ReY99] (Chapter II, Prop. 3.7) or [PiY03](p.298):

$$E\left[e^{-\frac{y^2}{2}T^{\gamma}_{-c,c}}\right] = \frac{1}{\cosh(yc)}, \tag{3.63}$$

we get:

$$\begin{aligned} E\left[\frac{1}{T^{\theta^{(1)}}_{-c,c}}\right] &= \int_0^{\infty} yE\left[e^{-\frac{y^2}{2}T^{\gamma}_{-c,c}}\right] \coth\left(\frac{\pi}{2}y\right)\ dy \\ &= \int_0^{\infty} \frac{y}{\cosh(yc)} \coth\left(\frac{\pi}{2}y\right)\ dy \equiv G(c). \end{aligned} \tag{3.64}$$

For $c = 2\pi$, we obtain the numerical result:

$$G(2\pi) \equiv E\left[\frac{1}{T^{\theta^{(1)}}_{-2\pi,2\pi}}\right] \approx 0.167. \tag{3.65}$$

Thus, with (3.62),

$$E\left[\ln\left(T^{\theta}_{-c,c}\right)\right] = 2\ln(\tilde{c}_n) + E\left[\ln\left(T^{\theta^{(1)}}_{-c,c}\right)\right], \tag{3.66}$$

and from (3.53), we have:

$$E\left[\ln\left(T^{\theta}_{-2\pi,2\pi} + \frac{1}{2}\right)\right] \leq 2\ln(\tilde{c}_n) + E\left[\ln\left(T^{\theta^{(1)}}_{-2\pi,2\pi}\right)\right] + \frac{1}{2\tilde{c}_n^2} E\left[\frac{1}{T^{\theta^{(1)}}_{-2\pi,2\pi}}\right]. \tag{3.67}$$

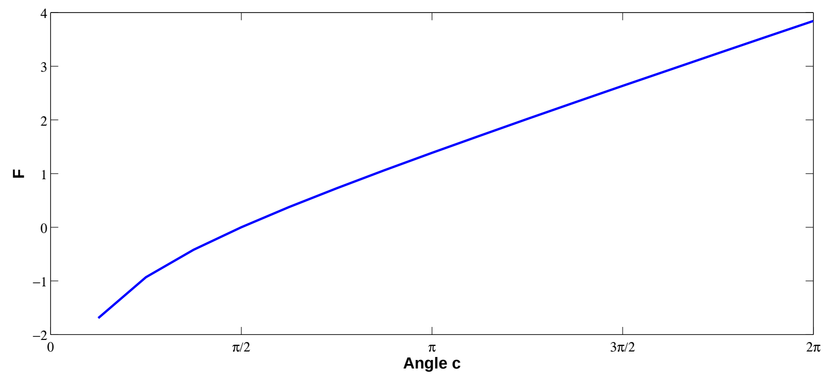


Figure 3.1: F **as a function of the angle** c**.**

In order to estimate the first moment of $\ln\left(T^{\theta^{(1)}}_{-c,c}\right)$, for an angle c, we follow section 2.1. From Prop. 2.1.11, there is the integral representation (2.51) (see also [CMY98]):

$$E\left[\ln\left(T^{\theta^{(1)}}_{-c,c}\right)\right] = 2F(c) + \ln(2) + c_E, \tag{3.68}$$

where[††]:

$$F(c) = \int_0^\infty \frac{dz}{\cosh\left(\frac{\pi z}{2}\right)} \ln\left(\sinh(cz)\right), \tag{3.69}$$

and $c_E \approx 0.577$ denotes Euler's constant.
For $c = 2\pi$, we have $F(2\pi) \approx 3.8$. In Figure 3.1 we plot F with respect to the angle c.
In summary, from (3.50), (3.65), (3.67) and (3.68) we approximate (actually, this is an upper bound for $E\left[T^{(\lambda)}_{2\pi}\right]$):

$$E\left[T^{(\lambda)}_{2\pi}\right] \approx \frac{1}{4}\left(2\ln(\tilde{c}_n) + Q + \frac{1}{2\tilde{c}_n^2} E\left[\frac{1}{T^{\theta^{(1)}}_{-2\pi,2\pi}}\right]\right), \tag{3.70}$$

where:

$$Q = 2F(2\pi) + 2\ln 2 + c_E \tag{3.71}$$

[††]We note that there is a simple relation between F and G: $\frac{\pi^2}{4c} F'\left(\frac{\pi^2}{4c}\right) = c\, G(c)$.

is a constant with $F(2\pi) \approx 3.8$, $c_E \approx 0.577$, and $E\left[\frac{1}{T^{\theta(1)}_{-2\pi,2\pi}}\right] \approx 0.167$, thus:

$$Q \approx 9.54. \tag{3.72}$$

Thus, by taking $c = 2\pi$, for n large, the mean time $E[\tilde{\tau}_n]$ that $\tilde{X}_n(t)$ rotates around $\mathbf{0}$, is:

$$E[\tilde{\tau}_n] \approx \frac{\sqrt{n}}{4}\left(2\ln(\tilde{c}_n) + \frac{0.08}{\tilde{c}_n^2} + Q\right), \tag{3.73}$$

and since:

$$\tilde{c}_n \equiv \frac{c_n}{\sqrt{n}}, \tag{3.74}$$

the MRT of $\tilde{X}_n(t)$ is given by the formula:

$$E[\tilde{\tau}_n] \approx \frac{\sqrt{n}}{4}\left[2\ln\left(\frac{c_n}{\sqrt{n}}\right) + 0.08\frac{n}{c_n^2} + Q\right]. \tag{3.75}$$

For a long enough polymer, such that $nl_0 >> L \Rightarrow n >> \tilde{l}$, thus $\tilde{l} = \frac{L}{l_0}$ is negligible with respect to $X_n(t)$ and from (3.34) $X_n(t) \approx \tilde{X}_n(t)$. For a mean initial configuration $c_n \equiv E\left(\sum_{k=1}^n e^{i\theta_k(0)}\right)$, we obtain that the MRT of the polymer is given by the formula:

$$E[\tau_n] \approx \frac{\sqrt{n}}{4}\left[2\ln\left(\frac{E\left(\sum_{k=1}^n e^{i\theta_k(0)}\right)}{\sqrt{n}}\right) + \frac{0.08n}{\left(E\left(\sum_{k=1}^n e^{i\theta_k(0)}\right)\right)^2} + Q\right]. \tag{3.76}$$

Finally, using the unscaled variables (3.5), we obtain:

$$E[\tau_n] \approx \frac{\sqrt{n}}{8D}\left[2\ln\left(\frac{E\left(\sum_{k=1}^n e^{\frac{i}{\sqrt{2D}}\theta_k(0)}\right)}{\sqrt{n}}\right) + \frac{0.08n}{\left(E\left(\sum_{k=1}^n e^{\frac{i}{\sqrt{2D}}\theta_k(0)}\right)\right)^2} + Q\right]. \tag{3.77}$$

Expressions (3.76) and (3.77) show that the leading order term of the MRT depends on the initial configuration, however this dependance is weak.

We consider now that the polymer is initially stretched ($\theta_k(0) = 0$, $\forall k = (1, \ldots, n)$), hence $c_n = n$. Thus, from (3.77), the MRT is approximately:

$$E[\tau_n] \approx \frac{\sqrt{n}}{8D}\left[\ln(n) + 0.08\frac{1}{n} + Q\right]. \tag{3.78}$$

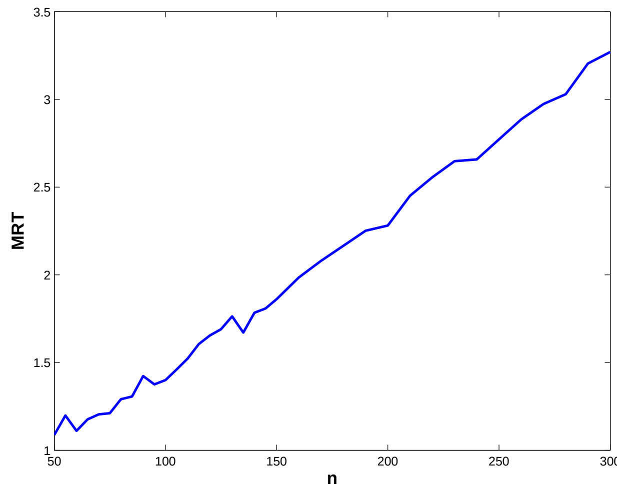


Figure 3.2: **MRT of the free polymer end as a function of the number of beads n (Brownian simulations).**

In order to check the range of validity of formula (3.78), we ran some Brownian simulations. In Figure 3.2, we simulated the MRT with a time step $dt = 0.01$ for $n = 50$ to 300 rods in steps of 5, and for each n, we took 300 samples and averaged over all of them. The parameters we chose are $D = 10$ for the diffusion coefficient, $L = 0.3$ for the distance from the origin $\mathbf{0}$ and $l_0 = 0.25$ for the length of each rod and for the initial condition we chose a stretched polymer located on the half line $\overrightarrow{\mathbf{0}x}$, $\theta_k(0) = 0$, $\forall k = (1, ..., n)$ (hence $c_n = L + nl_0$), and then we computed the MRT $E[\tau_n]$.

In Figure 3.3, we plot both the results from Brownian simulations and the formula (3.78). We considered values from $n = 50$ to 300 rods in steps of 10, $D = 10$, $L = 0.3$, $l_0 = 0.25$, thus the condition $\sqrt{n}\, l_0 >> L$ is satisfied. For each numerical computation, we performed 300 runs with a time step $dt = 0.01$. By comparing the numerical simulations (Fig. 3.3), with the analytical formula for the MRT, we see an overshoot.

The straight initial configuration is no restriction to the generality of our study: we have $c_n \leq n$ and the upper bound is achieved for a straight initial configuration. In Figure 3.4, we simulate the MRT $E[\tau_n]$ for both a random and an initially straight configuration (Brownian simulations with the same values for the parameters as above, after formula (3.78)).

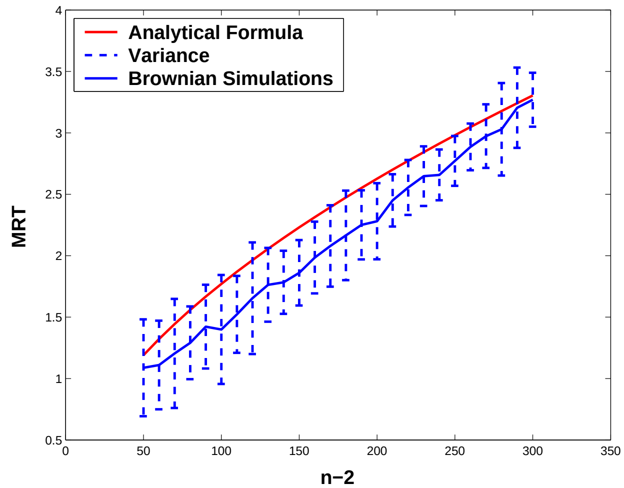


Figure 3.3: **Comparing the Analytical Formula and the Brownian simulations of the MRT.** The MRT of the polymer end is depicted with respect to the number of beads n.

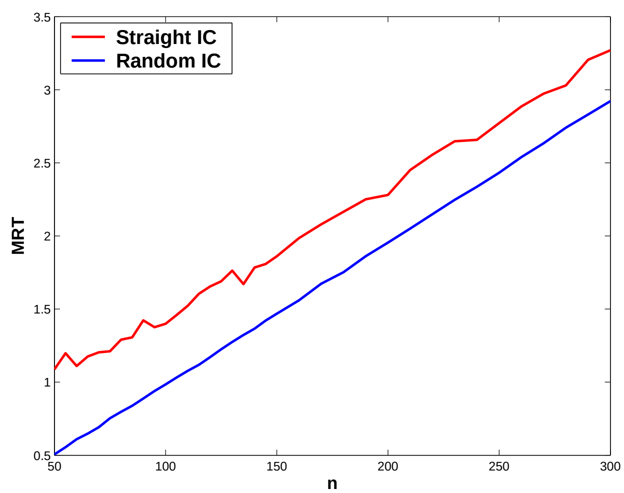


Figure 3.4: **MRT of the free polymer end as a function of the number of beads n for the straight and for a random initial configuration (Brownian simulations).**

Uniformly distributed initial angles

When the initial angles $(\theta_1(0), \theta_2(0), \ldots, \theta_n(0))$ are uniformly distributed over $[0, 2\pi]$, by averaging over all possible initial configurations, from (3.8) and (3.9), we obtain:

$$c_n = E\left[\sum_{k=1}^{n} e^{i\theta_k(0)}\right] = E\left[\sum_{k=1}^{n} (\cos(\theta_k(0)) + i\ \sin(\theta_k(0)))\right] = 0, \quad (3.79)$$

hence, from (3.17),

$$E\left[X_n(t)\right] \equiv \tilde{l} + c_n = \tilde{l}. \quad (3.80)$$

We define:

$$\hat{\tilde{X}}_n(t) \equiv \frac{1}{\sqrt{n}}\left(X_n(t) - \tilde{l}\right) + 1 = 1 + Z_t^{(n)}. \quad (3.81)$$

We know that, for n large, with $\hat{\tilde{\varphi}}_n(t)$ denoting the total angle of $\hat{\tilde{X}}_n(t)$ the mean time $E\left[\hat{\tilde{\tau}}_n\right] \equiv \inf\{t > 0, |\hat{\tilde{\varphi}}_n(t)| = 2\pi\}$ that it rotates around $\mathbf{0}$, is:

$$E\left[\hat{\tilde{\tau}}_n\right] \approx \frac{\sqrt{n}}{8D}\tilde{Q}.$$

Finally, for a long enough polymer, such that $nl_0 >> L \Rightarrow n >> \tilde{l}$ and $\sum_{k=1}^{n} e^{i\theta_k(t)} >> \sqrt{n}$, thus $\tilde{l} = \frac{L}{l_0}$ and $\sqrt{n}$ are negligible with respect to $X_n(t)$ and from (3.81) $X_n(t) \approx \hat{\tilde{X}}_n(t)$. By using the unscaled variables (3.5), with $\tilde{Q} \approx 9.62$, the MRT satisfies:

$$E\left[\tau_n\right] \approx E\left[\hat{\tilde{\tau}}_n\right] \approx \frac{\sqrt{n}}{8D}\tilde{Q}. \quad (3.82)$$

Remark 3.3.4 *Formula (3.77) or formulas (3.78) and (3.82) provide the asymptotic expansion for the MRT when $\theta_1 \in \mathbb{R}_+$. In fact, θ_1 is a reflected Brownian motion in $[0, 2\pi]$, thus a better characterization is to estimate the MRT by using the probability density function for θ_1 in the one dimensional torus $[0, 2\pi]$. In Appendix A.4, we derive this probability density function and by repeating the previous calculations, we show that, e.g. formula (3.82), remains valid.*

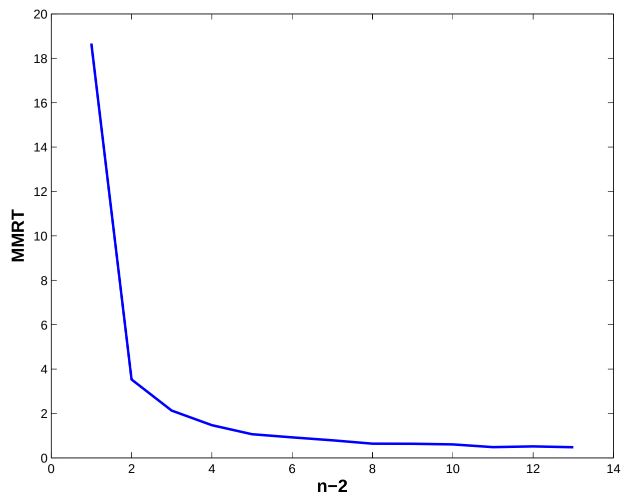


Figure 3.5: **Brownian simulation of the MMRT as a function of the number of beads $n-2$ for $D=10$.**

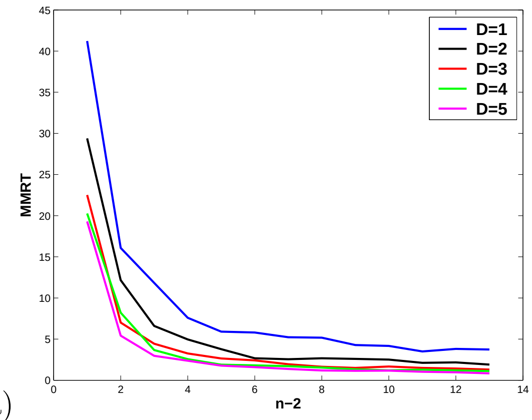
3.3.1 The Minimum Mean First Rotation Time

The Minimum Mean Rotation Time (MMRT) is the first time that any of the segments of the polymer loops around the origin,

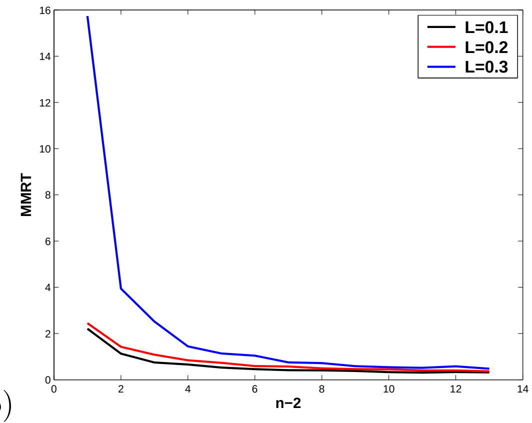
$$MMRT \equiv \min_{\mathcal{E}_n} E\{\tau_n|c_n\} \equiv E\left[\tau_{min}\right], \tag{3.83}$$

where $\mathcal{E}_n$ is the ensemble of rods which can travel up to the origin. The MMRT is now a decreasing function of n. In Figure 3.5, we present some simulations for the MMRT as a function of n (100 simulations per time step $dt = 0.01$ with $n = 4$ to 15 rods, $D = 10$ for the diffusion constant , $L = 0.3$ for the distance from the origin $\mathbf{0}$ and $l_0 = 0.25$ the length of each rod). The initial configuration is such that $|\varphi_n(0)| < 2\pi - \varepsilon$, e.g. the straight initial configuration: $\theta_k(0) = 0$, $\forall k = (1, ..., n)$.

In Figure 3.6, we present some Brownian simulations for the MMRT as a function of D and of L (Figure 3.6(a) and Figure 3.6(b) respectively). L and l_0 satisfy the rotation compatibility condition $nl_0 > L$. $E\left[\tau_{min}\right]$ decreases with D and increases with the distance from the origin L.



(a)



(b)

Figure 3.6: **Brownian simulation of the MMRT as a function of the number of beads $n-2$.** (a) For several values of D with $L = 0.3$, (b) for several values of L with $D = 10$.

3.3.2 Initial configuration in the boundary layer

When the polymer has initially almost made a loop, we expect the MRT to have a different behavior. We look at this numerically, and we start with an initial polymer configuration in the boundary layer :

$$2\pi - \varepsilon \leq |\varphi_n(0)| < 2\pi.$$

where φ_n is defined in eq. (3.4). The rotation of the polymer will be completed very fast and using Brownian simulations (100 runs per point with $n = 10$ rods and $D = 1$ for the diffusion constant, $L = 0.1$ for the distance from the origin $\mathbf{0}$ and $l_0 = 0.2$ for the length of each rod), we plotted in Figure 3.7 the results showing that when the initial total angle $\varphi_n(0)$ tends to 2π, the MRT tends to zero, with the precise asymptotic remaining to be completed. Numerically, we postulate that there is a threshold for an initial total angle $|\varphi_n(0)| = \frac{\pi}{2}$. When $|\varphi_n(0)| < \frac{\pi}{2}$, the MRT appears to be independent from this angle, whereas for $|\varphi_n(0)| > \frac{\pi}{2}$, the MRT decreases to zero. However, this needs further investigation.

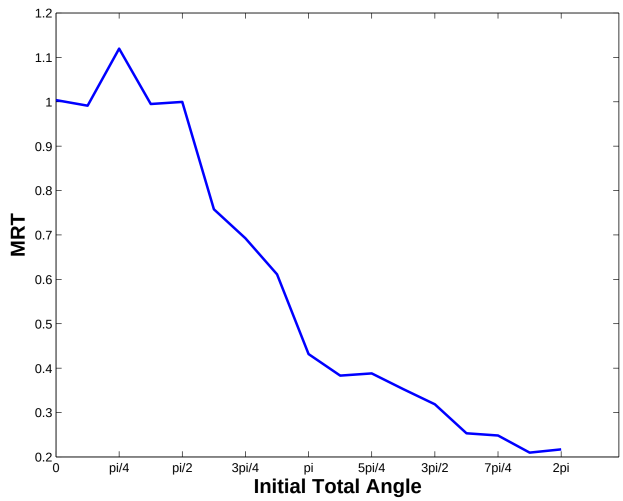


Figure 3.7: **Brownian simulations of the MRT as a function of the initial total angle $\varphi_n(0)$, where φ_n is defined in (3.4).**

Chapter 4

Conclusion and perspectives

Windings of stochastic processes and more specifically windings of Brownian motion has been a subject of study for many years [Spi58, PiY86] etc. In this PhD thesis, we investigate further the continuous winding process and we apply the results to study the polymer movement inside a cell.

In the third chapter of this PhD thesis, we are interested in the mean first time a polymer makes a loop around a fixed point. This is the first time that such problem is systematically investigated. To that purpose, we had to use and develop some tools coming from Probability theory and Stochastic Calculus. In the second chapter, we start studying the continuous winding process of the planar Brownian motion. Later on, we generalize our approach to complex valued Ornstein-Uhlenbeck processes. More specifically, we study the first hitting times defined by:

$$T^{\theta}_{-d,c} \equiv \inf\{t : \theta_t \notin (-d, c)\}, \ (c, d > 0),$$

To estimate $T^{\theta}_{-d,c}$, we shall use Bougerol's identity which says that, if $(\beta_u, u \geq 0)$ is a linear Brownian motion, for fixed $u > 0$,

$$\sinh(\beta_u) \overset{(law)}{=} \hat{\beta}_{\left(\int_0^u ds \exp(2\beta_s)\right)},$$

where $(\hat{\beta}_t, t \geq 0)$ is a Brownian motion, independent of β.

We find that:

$$\theta_{T_c^{\hat{\beta}}} \overset{(law)}{=} C_{a(c)},$$

where $\hat{\beta}$ is a Brownian motion independent of the process $(\theta_u, u \geq 0)$, $T_c^{\hat{\beta}} = \inf\{t > 0 : \hat{\beta}_t = c\}$, $(C_t, t \geq 0)$ is a standard Cauchy process and:

$$a(c) = \arg\sinh(c) \equiv \log\left(c + \sqrt{1+c^2}\right),\ c \in \mathbb{R}.$$

Our method allows to give in Proposition 2.1.7 a new non-computational proof of Spitzer's theorem [Spi58], which gives -for the Brownian motion- the asymptotic behavior of the winding number around the origin:

$$\frac{2}{\log t}\,\theta_t \underset{t\to\infty}{\overset{(law)}{\longrightarrow}} C_1\ ,$$

where C_1 is a standard Cauchy variable.

Moreover, we investigate further the planar Brownian motion by using Bougerol's identity and we study the distributions of the first hitting times $T^{\theta}_{-\infty,c}$ and $T^{\theta}_{-c,c}$. In particular, we give an explicit formula for the density function of the variable $T^{\theta}_{-c,c}$ and for the first moment of $\ln\left(T^{\theta}_{-c,c}\right)$. We give some analogues for a complex valued Ornstein-Uhlenbeck process with parameter λ, we exhibit the distribution of $T_c^{(\lambda)} = T_c^{\theta} = \inf\{t : |\theta_t^Z| = c\}$ and we derive the asymptotics of $E\left[T_c^{(\lambda)}\right]$ for λ large and for λ small. Finally, taking up again Bougerol's identity, with:

$$A_t(B) = \int_0^t ds \exp(2B_s),$$

where $(B_t, t \geq 0)$ is 1-dimensional Brownian motion, we show that $1/A_t(B)$, considered after a suitable measure change from Wiener measure, is infinitely divisible. More specifically, under a new measure $P_t = \left(\frac{t}{A_t(B)}\right)^{1/2} \cdot W$, where W is the Wiener measure, $1/A_t(B)$ is decomposed as the sum of N^2 and a subordinator $Y = (Y_t, t \geq 0)$ with $Y_0 = 0$ a.s., i.e. an increasing Lévy process, with N denoting a standard reduced Gaussian variable. Here, one can ask whether we can find an infinite divisibility property of $1/A_t(B)$ under the Wiener measure W. A possible approach may be through the Laplace transforms given in Corollary 2.1.3.

In conclusion, Bougerol's identity, which has been the main tool in this PhD thesis, calls for further investigation. The approach of this identity by using Laplace transforms (see e.g. Proposition 2.1.2 and equation (2.82)) that we proposed here, may give a new direction to research concerning some generalizations in higher dimensions. It is not clear how to extend Bougerol's identity to other processes than Brownian motion. We deal here only with an extension related to Ornstein-Uhlenbeck processes.

In the third chapter, we studied the MRT for a planar polymer consisting of n rods of length l_0. The first end is fixed at a distance L from the origin, while the other end moves as a Brownian motion. Interestingly, we have shown here that the motion of the free polymer end satisfies a new stochastic equation (3.34), containing a nonlinear time-dependent deterministic drift. When n is large, the limit stochastic equation is an Ornstein-Uhlenbeck process, with different time scales for each of the two coordinates.

We found that the MRT $E[\tau_n]$ actually depends on the mean initial configuration of the polymer. This result is in contrast with the small hole theory [WaK93, WHK93, HoS04, SSHE06, SSH06a, SSH06b, SSH07, SSH08] which shows that the leading order term of the MFPT for a Brownian particle to reach a small hole does not depend on the initial configuration. There are several reasons for such a difference. First, the position of the free moving end is a sum of i.i.d. variables and is not Markovian, which leads to a process with memory. Second, we may not be exactly in the context of a small hole, because the polymer completes a rotation when the free end reaches any point of the positive x-axis. In summary, for $nl_0 >> L$ and $n \geq 3$, the leading order term of the MRT is given by:

1. for a general initial configuration:

$$E[\tau_n] \approx \frac{\sqrt{n}}{8D}\left[2\ln\left(\frac{c_n}{\sqrt{n}}\right) + 0.08\frac{n}{c_n^2} + Q\right],$$

   where $c_n \equiv E\left(\sum_{k=1}^{n} e^{\frac{i}{\sqrt{2D}}\theta_k(0)}\right)$, $(\theta_k(0), 1 \leq k \leq n)$ is the sequence of the initial angles and $Q \approx 9.54$,

2. for a stretched initial configuration:

$$E[\tau_n] \approx \frac{\sqrt{n}}{8D}\left(\ln(n) + Q\right),$$

3. for an average over uniformly distributed initial angles:

$$E[\tau_n] \approx \frac{\sqrt{n}}{8D}\tilde{Q},$$

where $\tilde{Q} \approx 9.62$.

As we have shown, these formulas are in very good agreement with Brownian simulations (Fig. 3.3).

The mean time $E[\tau_{min}]$ when a loop is produced for the first time for the n rods also needs further investigation in order to provide an explicit formula. However, the simulations showed that $E[\tau_{min}]$ decreases with the polymer length nl_0, it also decreases with the diffusion constant D and increases with the distance from the origin L.

Finally, in order to get a better accuracy of the MRT, we consider that each angle θ_k between the rods is a reflected Brownian motion in the one-dimensional torus $[0, 2\pi]$ (Appendix A.4).

Concerning further research in the future, one can ask, instead of taking the two limits of $Z_t^{(n)}$, given in (3.25) separately (first for $n \to \infty$ and after for $t \to \infty$, see e.g. Corollary 3.2.3), what is its asymptotic behavior when $n, t \to \infty$ simultaneously? Moreover, it would also be interesting to use the series expansion of $\log(1 + x)$ in (3.52) in order to provide more accurate results. For this purpose, we should take into account all the negative moments of $T^{\theta}_{-c,c} = A_{T^{\gamma}_{-c,c}}(B) = \int_0^{T^{\gamma}_{-c,c}} ds \exp(2B_s)$.

Moreover, extension of this work should answer the following questions: Given that the polymer has already made one loop, what is the probability to make a second loop before unwrapping? What is the MRT in the case that a polymer moves in three dimensions? One can ask how long a polymer takes to wind around a one-dimensional filament, which can be used to study the motion of plasmid moving inside the cell cytoplasm. In a first approach, one can project the 3d motion into the following plane: we fix an end X_0 of the polymer and we define the plane which is perpendicular to the filament in the direction of the vector $\overrightarrow{\mathbf{0}X_0}$. Then, we can extend the 2d analysis developed here to study the projection. For example, in the 2-d projection the lengths of the different rods are not any more constant, but will vary with time. However, if we consider the angle of each rod with respect to the plane that we project as Brownian, we know that the mean length of each rod remains constant. Another possible starting point to study the 3d case may be via the three dimensional Brownian motion and its windings around a finite number of straight lines [LeGY87].

Finally, another possible direction would be to use a more sophisticated model for the polymer, e.g. the Rouse model or models that additionally include bending, torsion or other mechanical properties [Rou53, SSS80, DoE94].

Appendix A

Appendix

The Appendix consists of four parts.

In the first part A.1, we give a table with Bougerol's identity and several equivalent expressions.

In the second part A.2, we proceed to some explicit calculations in order to show that:

$$\begin{aligned}
&\Big(L(x) \equiv \log(\sqrt{x} + \sqrt{1+x}) \equiv a_s(\sqrt{x}),\ x \geq 0\Big) \\
&\qquad \text{and} \\
&\Big(\Phi(x) \equiv (\log(\sqrt{x} + \sqrt{1+x}))^2 \equiv (a_s(\sqrt{x}))^2,\ x \geq 0\Big)
\end{aligned}$$

are Bernstein functions.

In the third one A.3, we give the proof of Theorem 3.2.1. We use some arguments coming from Probability theory and we prove the convergence of the sequence $(M_t^{(n)}, t \geq 0)$ to a Brownian motion in dimension 2, with different time scale functions.

In the fourth part A.4, we first calculate the probability density function of the reflected Brownian angle θ_1 in the one-dimensional torus $[0, 2\pi]$. Second, we use this probability density function and we repeat the same arguments as in section 3.3 to provide a more accurate estimation of the MRT.

A.1 Table of Bougerol's Identity and other equivalent expressions

Bougerol's Identity and equivalent expressions $(u > 0$ fixed$)$
1) $\sinh(\beta_u) \overset{(law)}{=} \hat{\beta}_{(A_u(\beta) \equiv \int_0^u ds \exp(2\beta_s))}$ (Bougerol's Identity)
2) $\sinh(\bar{\beta}_u) \overset{(law)}{=} \bar{\hat{\beta}}_{(A_u(\beta))}, \quad \bar{\beta}_u = \sup_{s \leq u} \beta_s$
3) $\sinh(C_c) \overset{(law)}{=} \hat{\beta}_{(T_c^\theta)}, \quad c > 0$
4) $H_{T_b^{\hat{\beta}}} \overset{(law)}{=} T^{\beta}_{a(b)}, \quad a(x) = \arg\sinh(x), \quad b > 0$
5) $\theta_{T_b^{\hat{\beta}}} \overset{(law)}{=} C_{a(b)}$
6) $\bar{\theta}_{T_b^{\hat{\beta}}} \overset{(law)}{=} \lvert C_{a(b)} \rvert, \quad \bar{\theta}_u = \sup_{s \leq u} \theta_s$
7) $E\left[\frac{1}{\sqrt{2\pi T_c^\theta}} \exp\left(-\frac{x}{2T_c^\theta}\right)\right] = \frac{1}{\sqrt{1+x}} \frac{c}{\pi(c^2 + \log^2(\sqrt{x} + \sqrt{1+x}))}, \quad x \geq 0$
8) $E\left[\frac{1}{\sqrt{2\pi T_{-c,c}^\theta}} \exp(-\frac{x}{2T_{-c,c}^\theta})\right] = \left(\frac{1}{c}\right) \left(\frac{1}{\sqrt{1+x}}\right) \frac{1}{(\sqrt{1+x}+\sqrt{x})^\zeta + (\sqrt{1+x}-\sqrt{x})^\zeta}, \ x \geq 0, \ \zeta = \frac{\pi}{2c}$
9) $E\left[\frac{1}{\sqrt{2\pi A_u(\beta)}} \exp\left(-\frac{x}{2A_u(\beta)}\right)\right] = \frac{1}{\sqrt{2\pi u}} \frac{1}{\sqrt{1+x}} \exp\left(-\frac{(a(\sqrt{x}))^2}{2u}\right), \quad x \geq 0$
10) $\theta^{Z^\lambda}_{T_b^{(\lambda)}(U^\lambda)} \overset{(law)}{=} C_{a(b)}$ (OU version)

Notations:
$(Z_t = X_t + iY_t, t \geq 0)$: is a standard planar Brownian motion, starting from $1 + i0$,
$(\beta_t, \hat{\beta}_t,\ t \geq 0)$ are two independent Brownian motions,
$\theta_t = \mathrm{Im}(\int_0^t \frac{dZ_s}{Z_s}), t \geq 0$ is the continuous winding process,
$H_t = \int_0^t \frac{ds}{|Z_s|^2}$,
$(C_t, t \geq 0)$: denotes a standard Cauchy process,
$T_c^\beta \equiv \inf\{t : \beta_t \notin (-\infty, c)\}$,
$T_c^\theta \equiv \inf\{t : \theta_t \notin (-\infty, c)\}$,
$T_{-c,c}^\theta \equiv \inf\{t : \theta_t \notin (-c, c)\}$,
$(Z_t^\lambda, t \geq 0)$: is a complex valued OU process with parameter $\lambda \geq 0$, starting from a point different from 0,
$(U_t^\lambda, t \geq 0)$: is a real valued OU process with parameter $\lambda \geq 0$, independent from Z^λ, starting from a point different from 0, and
$T_b^{(\lambda)}(U^\lambda) \equiv \inf\left\{t \geq 0 : e^{\lambda t} U_t^\lambda = b\right\}$.

A.2 Two Bernstein functions

$\left(L(x) \equiv \log(\sqrt{x}+\sqrt{1+x}) \equiv a_s(\sqrt{x}),\ x \geq 0\right)$ as a Bernstein function.

We look for $\mu(da)$, a Lévy measure on $\mathbb{R}_+$, such that:

$$L(x) \equiv \log(\sqrt{x}+\sqrt{1+x}) = \int_0^\infty \mu(da)\ \left(1-e^{-xa}\right). \tag{A.1}$$

Taking derivatives on both sides, we get:

$$\frac{1}{\sqrt{x}+\sqrt{1+x}}\left\{\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{1+x}}\right\} \equiv \frac{1}{2\sqrt{x(1+x)}} = \int_0^\infty \mu(da)\, a\, e^{-xa}$$

We then write:

$$\begin{aligned}
\frac{1}{\sqrt{x(1+x)}} &= \frac{1}{\pi}\left(\int_0^\infty \frac{dt}{\sqrt{t}}\, e^{-xt}\right)\left(\int_0^\infty \frac{ds}{\sqrt{s}}\, e^{-(1+x)s}\right)\\
&\overset{u=t+s}{=} \frac{1}{\pi}\int_0^\infty \frac{ds}{\sqrt{s}}\, e^{-s}\int_s^\infty \frac{du}{\sqrt{u-s}}\, e^{-xu}\\
&\overset{s=hu}{=} \frac{1}{\pi}\int_0^\infty du\, e^{-xu}\left(\int_0^1 \frac{dh}{\sqrt{h(1-h)}}\, e^{-hu}\right), \qquad \text{(a)}
\end{aligned}$$

so that, finally, we have identified $\mu(da)$ in (A.1) as:

$$\mu(da) = \frac{da}{2a}E\left[e^{-ag}\right], \qquad \text{(b)}$$

where g is arcsine distributed on $[0,1]$.
We note that $\mu(da)$ is the Lévy measure of a GGC variable, which has been the subject of studies [BFRY06], followed by a more general discussion [JRY08]. We recall that:

$$\sqrt{1+x}-\sqrt{x} \equiv \frac{1}{\sqrt{1+x}+\sqrt{x}} = \exp\left(-L(x)\right)$$

is the Laplace transform of $d_{\mathbf{e}} - g_{\mathbf{e}}$, where:

$$d_t = \inf\left\{s \geq t : B_s = 0\right\}, \quad g_t = \sup\left\{s \leq t : B_s = 0\right\}$$

for a BM $(B_s, s \geq 0)$, and $\mathbf{e}$ is a standard exponential variable independent of B. An extension to Bessel processes with dimension $d < 2$ is given in [BFRY06]. Moreover, relying upon a private note by F. Hirsch [Hir10], we can obtain the Thorin measure [Tho77, JRY08] of $L(x)$. Indeed, if we consider $\tilde{L}(x)$ the GGC of Thorin measure $\frac{1}{2\pi}\frac{1}{\sqrt{t(1-t)}}1_{(0,1)}(t)$, a simple calculation and the change of variables $t = \frac{v^2}{1+v^2}$ yield:

$$\begin{aligned} \tilde{L}'(x) &= \frac{1}{2\pi}\int_0^1 \frac{1}{(x+t)\sqrt{t(1-t)}}\,dt = \frac{1}{\pi}\int_0^\infty \frac{dv}{x+(x+1)v^2} \\ &= \frac{1}{2}\frac{1}{\sqrt{x(1+x)}} = L'(x). \end{aligned} \tag{A.2}$$

Thus, $\tilde{L}(x) = L(x)$ is GGC with Thorin measure $\frac{1}{2\pi}\frac{1}{\sqrt{t(1-t)}}1_{(0,1)}(t)$.

$\left(\Phi(x) \equiv (\log(\sqrt{x}+\sqrt{1+x}))^2 \equiv (a_s(\sqrt{x}))^2,\ x \geq 0\right)$ as a Bernstein function.

We look for $\nu(da)$, a Lévy measure on $\mathbb{R}_+$, such that:

$$\Phi(x) \equiv \left(\log(\sqrt{x}+\sqrt{1+x})\right)^2 = \int_0^\infty \nu(da)\ \left(1-e^{-xa}\right). \tag{A.3}$$

Taking derivatives on both sides, we get:

$$2\log(\sqrt{x}+\sqrt{1+x})\frac{1}{2\sqrt{x(1+x)}} \equiv L(x)\frac{1}{\sqrt{x(1+x)}} = \int_0^\infty \nu(da)\, a\, e^{-xa}\ . \tag{A.4}$$

With the help of both (a) and (b), we can write the LHS of (A.4) as:

$$\Phi'(x) \equiv L(x)\frac{1}{\sqrt{x(1+x)}} = \int_0^\infty \frac{du}{u}E\left[e^{-ug}\right]\left(1-e^{-xu}\right)\int_0^\infty dv\ e^{-xv}E\left[e^{-vg}\right].$$

Similarly to the previous paragraph, relying upon a private note by F. Hirsch [Hir10], we can obtain the Thorin measure of $\Phi(x)$, which also means that $\Phi(x)$ is a Bernstein function. Indeed, if we consider $\tilde{\Phi}(x)$ the GGC of Thorin measure $\frac{1}{2}\frac{1}{\sqrt{t(t-1)}}1_{(1,\infty)}(t)$, a simple calculation and the change of variables $t = \frac{x}{x-(x+1)\tanh^2 v}$ yield:

$$\tilde{\Phi}'(x) = \frac{1}{2}\int_1^\infty \frac{dt}{(x+t)\sqrt{t(t-1)}} = \frac{1}{\sqrt{x(1+x)}} \arg\tanh \sqrt{\frac{x}{x+1}} \; .$$

However, by using the explicit expression: $\arg\tanh y = \frac{1}{2}\log\frac{1+y}{1-y}$, we deduce:

$$\tilde{\Phi}'(x) = \frac{1}{2}\frac{1}{\sqrt{x(1+x)}}\log(\sqrt{x}+\sqrt{1+x}) = \Phi'(x). \tag{A.5}$$

Thus, $\tilde{\Phi}(x) = \Phi(x)$ is GGC with Thorin measure $\frac{1}{2}\frac{1}{\sqrt{t(t-1)}}1_{(1,\infty)}(t)$, thus $\Phi(x)$ is a Bernstein function.

A.3 Proof of Theorem 3.2.1

To prove the theorem, we study each sequence $S_t^{(n)}$ and $C_t^{(n)}$ separately. The change of time comes from Dambis-Dubins-Schwarz Theorem and the scaling property of the Brownian motion [ReY99]. For a fixed time $t \geq 0$, we have:

$$\begin{aligned} C_t^{(n)} &:= \frac{1}{\sqrt{n}}\int_0^t \sum_{k=1}^n \cos(B_k(s))\, dB_k(s) \\ &= \frac{1}{\sqrt{n}}\beta^{(n)}_{\left(\int_0^t \sum_{k=1}^n \cos^2(B_k(s))\, ds\right)} \\ &= \tilde{\beta}^{(n)}_{\left(\frac{1}{n}\int_0^t \sum_{k=1}^n \cos^2(B_k(s))\, ds\right)} \quad , \end{aligned} \tag{A.6}$$

and:

$$\begin{aligned} S_t^{(n)} &:= \frac{1}{\sqrt{n}}\int_0^t \sum_{k=1}^n \sin(B_k(s))\, dB_k(s) \\ &= \frac{1}{\sqrt{n}}\hat{\beta}^{(n)}_{\left(\int_0^t \sum_{k=1}^n \sin^2(B_k(s))\, ds\right)} \\ &= \hat{\tilde{\beta}}^{(n)}_{\left(\frac{1}{n}\int_0^t \sum_{k=1}^n \sin^2(B_k(s))\, ds\right)} \quad , \end{aligned} \tag{A.7}$$

where $(\beta_u, \hat{\beta}_u, \tilde{\beta}_u, \hat{\tilde{\beta}}_u, u \geq 0)$ are four Brownian motions. To study the limit of $(S_u^{(n)}, C_u^{(n)}, u \geq 0)$ and prove convergence in law, we shall show [Bil68, Bil78] or [ReY99] (Chapter XIII):

1. The convergence of the finite dimensional distributions and
2. the tightness of the sequence $(M_t^{(n)}, t \geq 0)$, where by definition, a sequence $(M_t^{(n)}, t \geq 0)$ of random variable is tight if, for all $\varepsilon > 0$, there exists a compact subset K such that $P(M_t^{(n)} \in K) > 1 - \varepsilon$ for all $n \geq 1$.

1. We consider two bounded functions f, g of compact support and we study the characteristic function of the processes $(S_u^{(n)}, C_u^{(n)}, u \geq 0)$, $(\forall \lambda, \mu \in \mathbb{R})$ (λ and μ could be omitted ($\lambda = \mu = 1$) as they don't play any role in the proof. However, it seems better to use them so as to have the general form of the characteristic function as known).
We shall use the notation $< \cdot >$ for the quadratic variation: for a real-valued stochastic process N_t and for a sequence $\Delta_n \equiv \{0 < t_1 < \ldots < t_n = t\}$ of subdivisions of an interval $[0, t]$ such that the mass $|\Delta_n| \equiv \sup_k (t_{k+1} - t_k)$ goes to 0 with n, for every t, with $P - \lim$ denoting the Probability limit, the quadratic variation is:

$$< N >_t := < N, N >_t = P - \lim_{|\Delta_n| \to 0} \sum_{k=1}^{n} \left(N_{t_k} - N_{t_{k-1}} \right)^2 .$$

More generally, the quadratic covariation of two real-valued stochastic processes $N_t^{(1)}$ and $N_t^{(2)}$, when it exists, is defined by:

$$< N^{(1)}, N^{(2)} >_t = P - \lim_{|\Delta_n| \to 0} \sum_{k=1}^{n} \left(N_{t_k}^{(1)} - N_{t_{k-1}}^{(1)} \right) \left(N_{t_k}^{(2)} - N_{t_{k-1}}^{(2)} \right) .$$

In our computation below, $N^{(1)}$ and $N^{(2)}$ are two continuous martingales, for which the above limit exists.
We study now:

$$E\left[e^{i\left(\lambda \int_0^\infty f(u) dS_u^{(n)} + \mu \int_0^\infty g(u) dC_u^{(n)}\right)} \right] = E\left[e^{i\left(\lambda \int_0^\infty f(u) dS_u^{(n)} + \mu \int_0^\infty g(u) dC_u^{(n)}\right)} \times \right.$$
$$\left. \times e^{\frac{1}{2}\left\langle \lambda \int_0^\cdot f(s) dS_s^{(n)} + \mu \int_0^\cdot g(s) dC_s^{(n)} \right\rangle_\infty - \frac{1}{2}\left\langle \lambda \int_0^\cdot f(s) dS_s^{(n)} + \mu \int_0^\cdot g(s) dC_s^{(n)} \right\rangle_\infty} \right] . \quad \text{(A.8)}$$

We know that since we have an exponential martingale [ReY99]:

$$E\left[e^{i\left(\lambda \int_0^\infty f(u) dS_u^{(n)} + \mu \int_0^\infty g(u) dC_u^{(n)}\right) + \frac{1}{2}\left\langle \lambda \int_0^\cdot f(s) dS_s^{(n)} + \mu \int_0^\cdot g(s) dC_s^{(n)} \right\rangle_\infty} \right] = 1. \quad \text{(A.9)}$$

Moreover:

$$\left\langle \lambda \int_0^{\cdot} f(s) dS_s^{(n)} + \mu \int_0^{\cdot} g(s) dC_s^{(n)} \right\rangle_u =$$

$$\begin{aligned} = \; & \lambda^2 \int_0^u f^2(s)\, d\left\langle S^{(n)} \right\rangle_s + \mu^2 \int_0^u g^2(s)\, d\left\langle C^{(n)} \right\rangle_s \\ & + 2\lambda\mu \int_0^u f(s) g(s)\, d\left\langle S^{(n)}, C^{(n)} \right\rangle_s . \end{aligned} \tag{A.10}$$

Using the Law of Large Numbers,

$$\begin{aligned} \left\langle S^{(n)}, C^{(n)} \right\rangle_u &= \frac{1}{n} \int_0^u \sum_{k=1}^n \sin(B_k(s)) \cos(B_k(s))\, ds \\ &\xrightarrow[n\to\infty]{(P)} \int_0^u E\left[\sin(B_1(s)) \cos(B_1(s))\right]\, ds \\ &= 0. \end{aligned} \tag{A.11}$$

Indeed, by the symmetry property of the Brownian motion $B_1 \overset{(law)}{=} -B_1$, thus: $E\left[\sin(B_1(s)) \cos(B_1(s))\right] = E\left[\sin(-B_1(s)) \cos(-B_1(s))\right]$ which yields: $E\left[\sin(B_1(s)) \cos(B_1(s))\right] = 0$. Similarly, the Law of Large Numbers yields:

$$\begin{aligned} \left\langle S^{(n)} \right\rangle_u &= \frac{1}{n} \int_0^u \sum_{k=1}^n \sin^2(B_k(s))\, ds \\ &\xrightarrow[n\to\infty]{(P)} \int_0^u E\left[\sin^2(B_1(s))\right]\, ds, \end{aligned} \tag{A.12}$$

and:

$$\begin{aligned} \left\langle C^{(n)} \right\rangle_u &= \frac{1}{n} \int_0^u \sum_{k=1}^n \cos^2(B_k(s))\, ds \\ &\xrightarrow[n\to\infty]{(P)} \int_0^u E\left[\cos^2(B_1(s))\right]\, ds. \end{aligned} \tag{A.13}$$

However:

$$E\left[(\cos(B_1(s)) + i \sin(B_1(s)))^2\right] = E\left[\cos^2(B_1(s)) - \sin^2(B_1(s))\right],$$

as, once more by symmetry, $E\left[\sin(B_1(s))\cos(B_1(s))\right] = 0$, and (recall that $\forall\lambda,\quad E\left[e^{\lambda B(s)}\right] = e^{\frac{\lambda^2}{2}s}$):

$$E\left[\left(\cos(B_1(s)) + i\sin(B_1(s))\right)^2\right] = E\left[e^{2iB_1(s)}\right] = e^{-2s}.$$

Hence:

$$E\left[\cos^2(B_1(s))\right] - E\left[\sin^2(B_1(s))\right] = e^{-2s},$$

and:

$$E\left[\cos^2(B_1(s))\right] + E\left[\sin^2(B_1(s))\right] = 1.$$

We deduce:

$$E\left[\cos^2(B_1(s))\right] = \frac{1+e^{-2s}}{2} \tag{A.14}$$

$$E\left[\sin^2(B_1(s))\right] = \frac{1-e^{-2s}}{2}. \tag{A.15}$$

Thus:

$$\begin{aligned} \left\langle S^{(n)}\right\rangle_u &= \frac{1}{n}\int_0^u \sum_{k=1}^n \sin^2(B_k(s))\,ds \\ &\xrightarrow[n\to\infty]{(P)} \int_0^u E\left[\sin^2(B_1(s))\right]\,ds \\ &= \frac{1}{2}\int_0^u \left(1-e^{-2s}\right)\,ds, \end{aligned} \tag{A.16}$$

and:

$$\begin{aligned} \left\langle C^{(n)}\right\rangle_u &= \frac{1}{n}\int_0^u \sum_{k=1}^n \cos^2(B_k(s))\,ds \\ &\xrightarrow[n\to\infty]{(P)} \int_0^u E\left[\cos^2(B_1(s))\right]\,ds \\ &= \frac{1}{2}\int_0^u \left(1+e^{-2s}\right)\,ds. \end{aligned} \tag{A.17}$$

Hence, from (A.8):

$$E\left[e^{i\left(\lambda\int_0^\infty f(u)dS_u^{(n)}+\mu\int_0^\infty g(u)dC_u^{(n)}\right)}\right]$$

$$\stackrel{n\to\infty}{\longrightarrow} \exp\left(-\frac{\lambda^2}{2}\int_0^\infty f^2(u)\frac{1}{2}\left(1-e^{-2u}\right)\,du-\frac{\mu^2}{2}\int_0^\infty g^2(u)\frac{1}{2}\left(1+e^{-2u}\right)\,du\right)$$

$$= E\left[e^{i\lambda\int_0^\infty f(u)dS_u}\right]E\left[e^{i\mu\int_0^\infty g(u)dC_u}\right], \tag{A.18}$$

which shows the convergence of the finite distributions.
2. In order to prove tightness, we shall use Kolmogorov's criterion: $(M_t^{(n)}, t\geq 0)$ is tight if, for every n, there exist positive constants α,β and c_1 such as:

$$E\left[\left|M_t^{(n)}-M_s^{(n)}\right|^\beta\right]\leq c_1\left|t-s\right|^{1+\alpha}. \tag{A.19}$$

Indeed, Cauchy-Schwarz inequality yields:

$$\begin{aligned} E\left[\left|M_t^{(n)}-M_s^{(n)}\right|^\beta\right] &= E\left[\left|S_t^{(n)}+iC_t^{(n)}-S_s^{(n)}-iC_s^{(n)}\right|^\beta\right]\\ \leq\ & 2^{\beta-1}\left\{\left(E\left[\left|S_t^{(n)}-S_s^{(n)}\right|^{2\beta}\right]\right)^{1/2}+\left(E\left[\left|C_t^{(n)}-C_s^{(n)}\right|^{2\beta}\right]\right)^{1/2}\right\}. \end{aligned} \tag{A.20}$$

By using Burkholder-Davis-Gundy inequality (BDG) [KaSh88, ReY99], there exists a constant $0<c_2<\infty$ such that:

$$E\left[\left|S_t^{(n)}-S_s^{(n)}\right|^{2\beta}\right]\leq c_2E\left[\left(\left\langle S^{(n)}\right\rangle_t-\left\langle S^{(n)}\right\rangle_s\right)^\beta\right]. \tag{A.21}$$

We have that:

$$\left\langle S^{(n)}\right\rangle_t-\left\langle S^{(n)}\right\rangle_s=\frac{1}{n}\int_s^t\sum_{k=1}^n\underbrace{\sin^2(B_k(u))}_{\leq 1}\,du\leq t-s. \tag{A.22}$$

Thus,

$$\left(E\left[\left|S_t^{(n)} - S_s^{(n)}\right|^{2\beta}\right]\right)^{1/2} \leq c_2^{1/2}\,|t-s|^{\beta/2}\,. \tag{A.23}$$

Similar computations for $C^{(n)}$, lead to a constant $0 < c_3 < \infty$ such that:

$$\left(E\left[\left|C_t^{(n)} - C_s^{(n)}\right|^{2\beta}\right]\right)^{1/2} \leq c_3^{1/2}\,|t-s|^{\beta/2}\,. \tag{A.24}$$

From (A.20), (A.23) and (A.24) we deduce (A.19) for e.g. $\alpha = 1$, $\beta = 4$ and $c_1 = 2^3(c_2^{1/2} + c_3^{1/2})$. So, we proved that $(M_t^{(n)}, t \geq 0)$ is tight.
Summarizing our results 1 and 2, we deduce that the sequences $(S_t^{(n)}$ and $C_t^{(n)}, t \geq 0)$ are asymptotically independent and finally,

$$(S_t^{(n)}, C_t^{(n)}, t \geq 0) \xrightarrow[n\to\infty]{(law)} (\hat{\tilde{\beta}}_{\left(\frac{1}{2}\int_0^t ds\,(1-e^{-2s})\right)}, \tilde{\beta}_{\left(\frac{1}{2}\int_0^t ds\,(1+e^{-2s})\right)}, t \geq 0).$$

$\square$

A.4 The density of θ_1

We should also take into account the fact that θ_1 is a reflected Brownian motion in $[0, 2\pi]$. This means that the equations (A.14) and (A.15) change as the density function $p(\theta, t)$ of θ_1 is different from the density function of the Brownian motion. In order to find the density function $p(\theta, t)$ of θ_1, we have to solve the following system [Gar64, Kni81]. It is the heat (diffusion) equation with some boundary conditions (BC) due to the periodicity as $\theta_1 \in [0, 2\pi]$:

$$\begin{cases} \frac{\vartheta}{\vartheta t}p(\theta,t) = D\frac{\vartheta^2}{\vartheta\theta^2}p(\theta,t) \\ p(0,t) = p(2\pi,t) \\ \frac{\vartheta p}{\vartheta\theta}(0,t) = \frac{\vartheta p}{\vartheta\theta}(2\pi,t), \end{cases} \tag{A.25}$$

and the initial condition (IC) $p(\theta, 0) \equiv \frac{1}{2\pi}$ (thus, from (3.7) $c_n = 0$). The solution of this equation is given by the form:

$$p(\theta,t) = \sum_{n=0}^{\infty} Ae^{-D\lambda_n t}\left[a_n\cos(\sqrt{\lambda_n}\theta) + b_n\sin(\sqrt{\lambda_n}\theta)\right], \tag{A.26}$$

where $\sqrt{\lambda_n} = n\frac{\pi}{2\pi} \Rightarrow \lambda_n = \frac{n^2}{4}$ and A, a_n and b_n are constants which are going to be defined by the (BC) and the (IC). Hence:

$$p(\theta,t) = A\sum_{n=0}^{\infty} e^{-\frac{Dn^2t}{4}}\left[a_n\cos(\frac{n\theta}{2}) + b_n\sin(\frac{n\theta}{2})\right]. \tag{A.27}$$

Following section 3.1, we scale $\tilde{t} = \frac{t}{2D}$ and $\tilde{l} = \frac{L}{l_0}$ (we use t for $\tilde{t}$). From the (BC), we have:

$$\begin{cases} a_n\cos(n\pi) + b_n\sin(n\pi) = a_n \\ -a_n\sin(n\pi) + b_n\cos(n\pi) = b_n \end{cases} \tag{A.28}$$

$$\Rightarrow \begin{cases} a_n(\cos(n\pi) - 1) = 0 \\ b_n(\cos(n\pi) - 1) = 0. \end{cases} \tag{A.29}$$

This system has a solution not identically 0, i.e. $(a_n, b_n) = (0,0)$, if and only if:

$$\cos(n\pi) = 1 \Rightarrow n : \text{even} \Rightarrow n = 2k,\ k \in \mathbb{Z}. \tag{A.30}$$

Thus, $\lambda_n = \frac{n^2}{4} = k^2$ and by replacing $\tilde{a}_n = Aa_n$ and $\tilde{b}_n = Ab_n$, (A.27) becomes:

$$p(\theta,t) = \sum_{n=0}^{\infty} e^{-\frac{n^2t}{2}}\left[\tilde{a}_n\cos(n\theta) + \tilde{b}_n\sin(n\theta)\right]. \tag{A.31}$$

For $t = 0$, by using the (IC) of the heat equation (A.25), we have:

$$p(\theta,0) \equiv \frac{1}{2\pi} = \sum_{n=0}^{\infty}\left[\tilde{a}_n\cos(n\theta) + \tilde{b}_n\sin(n\theta)\right]. \tag{A.32}$$

Hence, $\tilde{a}_0 = \frac{1}{2\pi}$ and $\tilde{a}_n = \tilde{b}_n = 0$, for every $n \geq 1$. Thus:

$$p(\theta,t) = \frac{1}{2\pi}, \tag{A.33}$$

and from (A.31), because all the terms in the sum are equal to 0 except for the one for $n = 0$:

$$\begin{aligned} E\left[\sin^2(\theta_1)\right] &= E\left[\frac{1-\cos(2\theta_1)}{2}\right] \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1-\cos(2\theta)}{2}\right)d\theta = \frac{1}{2}. \end{aligned} \tag{A.34}$$

By repeating the same calculation, we have:

$$\begin{aligned} E\left[\cos^2(\theta_1)\right] &= E\left[\frac{1+\cos(2\theta_1)}{2}\right] \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1+\cos(2\theta)}{2}\right)d\theta = \frac{1}{2}. \end{aligned} \tag{A.35}$$

Equation (3.17) yields:

$$E\left(X_n(t)\right) = \tilde{l},$$

and we take up again our study, starting from equations (3.19) and (3.21). Using the Law of Large Numbers, the equations (A.16) and (A.17) change to:

$$\begin{aligned} \left\langle S^{(n)}\right\rangle_u &= \frac{1}{n}\int_0^u \sum_{k=1}^n \sin^2(\theta_k(s))\,ds \\ &\underset{n\to\infty}{\xrightarrow{(P)}} \int_0^u E\left[\sin^2(\theta_1(s))\right]\,ds \\ &= \int_0^u \frac{1}{2}\,ds = \frac{u}{2}, \end{aligned} \tag{A.36}$$

and:

$$\begin{aligned} \left\langle C^{(n)}\right\rangle_u &= \frac{1}{n}\int_0^u \sum_{k=1}^n \cos^2(\theta_k(s))\,ds \\ &\underset{n\to\infty}{\xrightarrow{(P)}} \int_0^u E\left[\cos^2(\theta_1(s))\right]\,ds \\ &= \int_0^u \frac{1}{2}\,ds = \frac{u}{2}. \end{aligned} \tag{A.37}$$

Following subsection 3.3, with $(\gamma_u, \tilde{\gamma}_u, u \geq 0)$ two independent Brownian motions:

$$Z_t^{(n)} = e^{-\frac{t}{2}}\int_0^t e^{\frac{s}{2}}dM_s^{(n)} \tag{A.38}$$

$$\underset{n\to\infty}{\xrightarrow{(law)}} e^{-\frac{t}{2}}\int_0^t \frac{e^{\frac{s}{2}}}{\sqrt{2}}\,d\gamma_s + ie^{-\frac{t}{2}}\int_0^t \frac{e^{\frac{s}{2}}}{\sqrt{2}}\,d\tilde{\gamma}_s \equiv Z_t^{(\infty)}. \tag{A.39}$$

Hence, by using the scaling property of Brownian motion:

$$Z_t^{(\infty)} = e^{-\frac{t}{2}} \int_0^t e^{\frac{s}{2}} \underbrace{\left(d\gamma_{s/2} + i\, d\tilde{\gamma}_{s/2}\right)}_{d\hat{\mathbb{B}}_{s/2}},$$

where $\left(\hat{\mathbb{B}}_t = \gamma_t + i\tilde{\gamma}_t, t \geq 0\right)$ is a 2-dimensional Brownian motion starting from 1. Consequently, by repeating the same arguments as in subsection 3.3, formula (3.82) remains valid.

Bibliography

[AbSt70] M. Abramowitz and I.A. Stegun (1970). Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables. Dover Publications, Inc., New York.

[ADY97] L. Alili, D. Dufresne, and M. Yor (1997). Sur l'identité de Bougerol pour les fonctionnelles exponentielles du mouvement Brownien avec drift. In *Exponential Functionals and Principal Values related to Brownian Motion. A collection of research papers; Biblioteca de la Revista Matematica, Ibero-Americana*, ed. M. Yor, 3-14.

[AKH11] A. Amitai, I. Kupka and D. Holcman (2011). Mean first looping time of a rigid rod polymer. In Preparation.

[And87] D. André (1887). Solution directe du problème résolu par M. Bertrand. *C. R. Acad. Sci. Paris*, **105**, 436-437.

[BeF91] C. Bélisle and J. Faraway (1991). Winding angle and maximum winding angle of the two-dimensional random walk. *J. Appl. Probab.*, **28**, 717-726.

[Bel86] C. Bélisle (1989). Limit distribution of winding of planar of random walks. *PhD dissertation*, Dept. Statistics, Univ. California, Berkeley.

[Bel89] C. Bélisle (1989). Windings of random walks. *Ann. Probab.*, **17**, 1377-1402.

[BeW94] J. Bertoin and W. Werner (1994). Asymptotic windings of planar Brownian motion revisited via the Ornstein-Uhlenbeck process. *Sém. Prob. XXVIII, Lect. Notes in Mathematics,* **1583**, Springer, Berlin Heidelberg New York 138-152.

[BFRY06] J. Bertoin, T. Fujita, B. Roynette and M. Yor (2006). On a particular class of self-decomposable random variables: the duration of a Bessel excursion straddling an independent exponential time. *Prob. Math. Stat.*, **26**, 315-366.

[Bil68] P. Billingsley (1968). Convergence of Probability Measures. John Wiley and Sons, New York.

[Bil78] P. Billingsley (1978). Probability and Measure. John Wiley and Sons, New York.

[BiY87] P. Biane and M. Yor (1987). Valeurs principales associées aux temps locaux browniens. *Bull. Sci. Math.*, **111**, 23-101.

[BKH07] A. Biess, E.Korkotian and D. Holcman (2007). Diffusion in a dendritic spine: The role of geometry. *Phys. Rev. E.*, **76**, 1.

[Bon90] L. Bondesson (1990). Generalized gamma convolutions and complete monotonicity. *Probab. Theory Related Fields*, **85**, 181-194.

[Bon92] L. Bondesson (1992). Generalized gamma convolutions and related classes of distributions and densities. *Lecture Notes in Statistics*, **76**. Springer-Verlag, New York.

[Bou83] Ph. Bougerol (1983). Exemples de théorèmes locaux sur les groupes résolubles. *Ann. Inst. H. Poincaré*, **19**, 369-391.

[BPY03] V. Bentkus, G. Pap, and M. Yor (2003). Optimal bounds for Cauchy approximations for the winding distribution of planar Brownian motion. *J. Theor. Probab.,* **16(2)**, ISSN 0894-9840, p. 259-279.

[Bur77] D. Burkholder (1977). Exit times of Brownian Motion, Harmonic Majorization and Hardy Spaces. *Adv. in Math.*, **26**, 182-205.

[CDM93] A. Comtet, J. Desbois and C. Monthus (1993). Asymptotic laws for the winding angles of planar Brownian motion. *J. Statist. Phys.*, **73**, 433-440.

[ChY03] L. Chaumont and M. Yor (2003). Exercises in Probability: A Guided Tour from Measure Theory to Random Processes, via Conditioning, Cambridge University Press.

[CMY98] A. Comtet, C. Monthus, and M. Yor (1998). Exponential functionals of Brownian motion and disordered systems. *J. Appl. Prob.*, **35**, 255-271.

[Com06] A. Comtet (2006). Personal Communication.

[D-MMY00] C. Donati-Martin, H. Matsumoto, and M. Yor (2000). On positive and negative moments of geometric Brownian motion. *Statistics and Probability Letters*, **49**, 45-52.

[D-MMY02] C. Donati-Martin, H. Matsumoto, and M. Yor (2002). Some absolute continuity relationships for certain anticipative transformations of geometric Brownian motion. *Publ. RIMS Kyoto Univ.*, **37**, 295-326.

[DoE94] M. Doi and S. F. Edwards (1994). The Theory of Polymer Dynamics. Oxford University Press.

[DrK96] B. Drossel and M. Kardar (1996). Winding Angle Distributions for Directed Polymers. In arXiv cond-mat/9610019 (October 1996).

[Duf00] D. Dufresne (2000). Laguerre Series for Asian and Other Options. *Mathematical Finance*, Vol. **10**, No. 4, 407-428.

[Dur82] R. Durrett (1982). A new proof of Spitzer's result on the winding of 2-dimensional Brownian motion. *Ann. Prob.* **10**, 244-246.

[DuY11] D. Dufresne and M. Yor (2011). A two-dimensional extension of Bougerol's identity in law for the exponential functional of Brownian motion: the story so far... In preparation.

[Gal08] L. Gallardo (2008). Mouvement Brownien et calcul d'Itô. Hermann.

[Gar64] P. Garabedian (1964). Partial Differential Equations, Wiley, New York.

[GDL11] D. Gredat, I. Dornic and J.M. Luck (2011). In ArXiv cond-mat.stat-mech/1101.1173 (January 2011).

[GrR65] I.S. Gradshteyn and I.M. Ryzhik (1965). Table of Integrals, Series and Products. Academic Press.

[Hir10] F. Hirsch (2010). Personal Communication.

[HoS04] D. Holcman and Z. Schuss (2004). Escape through a small opening: receptor trafficking in a synaptic membrane, *J. of Statistical Physics*, **117**, (5/6) 191-230.

[Ish05] K. Ishiyama (2005). Methods for Evaluating Density Functions of Exponential Functionals Represented as Integrals of Geometric Brownian Motion. *Methodology and Computing in Applied Probability*, Springer **7**, 271-283.

[ItMK65] K. Itô and H.P. McKean (1965). Diffusion Processes and their Sample Paths. Springer, Berlin Heidelberg New York.

[Itô83] K. Itô (1983).Distribution-Valued Processes arising from Independent Brownian Motions. *Mathematische Zeitschrift*, **182**, 17-33. Springer-Verlag.

[JRY08] L.F. James, B. Roynette and M. Yor (2008). Generalized Gamma Convolutions, Dirichlet means, Thorin measures, with explicit examples. *Probab. Surveys*, Volume **5** , 346-415.

[KaSh88] I. Karatzas and S.E. Shreve (1988). Brownian Motion and Stochastic Calculus. Berlin Heidelberg New York.

[Kni81] F.B. Knight (1981). Essentials of Brownian motion and diffusion. *Math. Surv.*, **18**, Amer. Math. Soc., Providence, Rhode-Island.

[LDEH09] P. Le Doussal, Y. Etzioni and B. Horovitz (2009). Winding of planar Gaussian processes. *J. Stat. Mech.: Theory and Experiment 2009*.

[Leb63] N.N. Lebedev (1963). Special Functions and their Applications. Dover Publications, Inc., New York.

[Leb72] N.N. Lebedev (1972). Special Functions and their Applications. Revised edition, translated from the Russian and edited by Richard A. Silverman.

[LeG90] J.F. Le Gall (1990-1992). Some properties of planar Brownian motion. *Ecole d'été de Saint-Flour XX*, 1990, *Lect. Notes in Mathematics*, Springer, Berlin Heidelberg New York 1992, **1527**, 112-234.

[LeGY87] J.F. Le Gall and M. Yor (1987). Etude asymptotique des enlacements du mouvement brownien autour des droites de l'espace. *Prob. Th. Rel. Fields* **74**, 617-635.

[Lev80] D. Dugué (1980). Œuvres de Paul Lévy, Vol. IV, Processus Stochastiques, Gauthier-Villars. **158** Random Functions: General Theory with Special Reference to Laplacian Random Functions by Paul Lévy.

[MaY98] H. Matsumoto and M. Yor (1998). On Bougerol and Dufresne's identities for exponential Brownian functionals. *Proc. Japan Acad. Ser. A Math. Sci.*, Volume **74**, Number 10, 152-155.

[MaY05] H. Matsumoto and M. Yor (2005). Exponential functionals of Brownian motion, I: Probability laws at fixed time. *Probab. Surveys* Volume **2**, 312-347.

[MaY08] R. Mansuy and M. Yor (2008). Aspects of Brownian Motion. Springer-Verlag, Berlin Heidelberg.

[MeY82] P. Messulam and M. Yor (1982). On D. Williams' "pinching method" and some applications. *J. London Math. Soc.*, **26**, 348-364.

[MNY02] H. Matsumoto, L. Nguyen, and M. Yor (2002). Subordinators related to the exponential functionals of Brownian bridges and explicit formulae for the semigroup of hyperbolic Brownian motions. *Stochastic Processes and Related Topics*, 213-235, Proc. 12th Winter School at Siegmundsburg (Germany), R.Buckdahn, H-J.Engelbert and M.Yor, eds., Stochastic Monographs, Vol. **12**, Taylor and Francis.

[Ochi85] Y. Ochi (1985). Limit theorems for a class of Diffusion Processes. *Stochastics* **15**, 251-269.

[PaY00] G. Pap and Yor, M. (2000). The accuracy of Cauchy approximation for the windings of planar Brownian motion. *Periodica Mathematica Hungarica* **41**, 213-226.

[PiY86] J.W. Pitman and M. Yor (1986). Asymptotic Laws of planar Brownian Motion. *Ann. Prob.* **14**, 733-779.

[PiY03] J.W. Pitman and M. Yor (2003). Infinitely divisible laws associated with hyperbolic functions. *Canad. J. Math.* **55**, 292-330.

[PWPSK09] S. Pillay, M. J. Ward, A. Pierce, R. Straube and T. Kolokolnikov (2009). Analysis of the mean first passage time for narrow escape problems, SIAM *J. Multiscale Modeling.*

[PZS96] R.W. Pastor, R. Zwanzig and A. Szabo (1996). Diffusion limited first contact of the ends of a polymer: Comparison of theory with simulation. *J. Chem. Phys.* **105**, 3878.

[ReY99] D. Revuz and M. Yor (1999). Continuous Martingales and Brownian Motion. 3rd ed., Springer, Berlin.

[Rou53] P. E. Rouse (1953). A Theory of the Linear Viscoelastic Properties of Dilute Solutions of Coiling Polymers. *J. Chem. Phys.*, **21**, 1272.

[RuH87] J. Rudnick and Y. Hu (1987). The winding angle distribution of an ordinary random walk. *J. Phys. A: Math. Gen.*, **20**, No. 13, 4421-4438.

[RuH88] J. Rudnick and Y. Hu (1988). Winding angle of a self-avoiding random walk. *Phys. Rev. Lett.*, **60**, No. 8, 712-715.

[Sal94] H. Saleur (1994). Winding-angle distribution for Brownian and self-avoiding walks. *Phys. Rev. E Stat. Phys. Plasmas Fluids Relat Interdiscip Topics.* **50**, 1123-1128.

[Shi98] Z. Shi (1998). Windings of Brownian motion and random walks in the plane. *The Annals of Probability*, **26**, No. 1, 112-131.

[Spi58] F. Spitzer (1958). Some theorems concerning two-dimensional Brownian Motion. *Trans. Amer. Math. Soc.* **87**, 187-197.

[Spi64] F. Spitzer (1964). Principles of random walk. V. Nostrand, Princeton, New Jersey.

[SSHE06] A. Singer, Z. Schuss, D. Holcman and B. Eisenberg (2006). Narrow Escape I. *J. Stat. Phys.* **122** (3), pp.437–463 .

[SSH06a] A. Singer, Z. Schuss and D. Holcman (2006). Narrow Escape II. *J. Stat. Phys.* **122** (3), pp.465–489.

[SSH06b] A. Singer, Z. Schuss and D. Holcman (2006). Narrow Escape III. *J. Stat. Phys.* **122** (3), pp.491-509.

[SSH07] Z. Schuss, A. Singer and D. Holcman (2007). Narrow Escape: Theory and Applications to Cellular Microdomains. *Proc. Nat. Acad. Sci.*, **41**, 104, 16098-16103.

[SSH08] A. Singer, Z. Schuss and D. Holcman (2008). Narrow escape and leakage of Brownian particles. *Physical Review E* **78**, 051111.

[SSS80] A. Szabo, K. Schulten and Z. Schulten (1980). First passage time approach to diffusion controlled reactions. *J. Chem. Phys.*, **72**, 4350-4357.

[Tho77] O. Thorin (1977). On the infinite divisibility of the lognormal distribution. *Scand. Actuar. J.*, **3**, 121-148.

[Vak10] S. Vakeroudis (2010). On hitting times of the winding processes of planar Brownian motion and of Ornstein-Uhlenbeck processes, via Bougerol's identity. Submitted to SIAM TVP Journal.

[Vak11] S. Vakeroudis (2011). On some infinite divisibility properties related with Bougerol's identity. In Preparation.

[Val92] P. Vallois (1992). Amplitude du mouvement Brownien et juxtaposition des excursions positives et négatives. *Sém. Prob. XXVI, Lect. Notes in Mathematics,* Springer-Verlag, p. 361-373.

[VYH11] S. Vakeroudis, M. Yor and D. Holcman (2011). The Mean First Rotation Time of a planar polymer. Accepted to Journal of Statistical Physics.

[WaK93] M.J. Ward and J.B. Keller (1993). Strong Localized Perturbations of Eigenvalue Problems. *SIAM J. Appl. Math.* **53**, pp.770–798.

[WHK93] M.J. Ward, W.D. Henshaw and J.B. Keller (1993). Summing Logarithmic Expansions for Singularly Perturbed Eigenvalue Problems. *SIAM J. Appl. Math.*, **53**, pp.799–828.

[Wie84] F.W. Wiegel (1984). Distribution of the angle of rotation for plane random walks. *Physica A: Theoretical and statistical physics*, **129** (1). pp. 211-219.

[WiF74] G. Wilemski and M.J. Fixman (1974). Diffusion-controlled intra-chain reactions of polymers. I Theory. *J. Chem. Phys.*, **60**, 866.

[Wil74] D. Williams (1974). A simple geometric proof of Spitzer's winding number formula for 2-dimensional Brownian motion. University College, Swansea. Unpublished.

[Yor80] M. Yor (1980). Loi de l'indice du lacet Brownien et Distribution de Hartman-Watson. *Z. Wahrsch. verw. Gebiete*, **53**, 71-95.

[Yor85] M. Yor (1985). Une décomposition asymptotique du nombre de tours du mouvement brownien complexe. [An asymptotic decomposition of the winding number of complex Brownian motion]. *Colloquium in honor of Laurent Schwartz*, Vol. **2** (Palaiseau, 1983). *Astérisque* No. **132** (1985), 103–126.

[Yor97] M. Yor (1997). Generalized meanders as limits of weighted Bessel processes, and an elementary proof of Spitzer's asymptotic result on Brownian windings. *Studia Scient. Math. Hung.* **33**, 339-343.

[Yor01] M. Yor (2001). Exponential Functionals of Brownian Motion and Related Processes. Springer Finance. Springer-Verlag, Berlin.